

QUANTUM THEORY
AND
LOCAL HIDDEN VARIABLE THEORY:
GENERAL FEATURES AND TESTS
FOR
EPR STEERING
AND
BELL NON-LOCALITY

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1 Abstract

Quantum states for bipartite composite systems are categorised as either separable or entangled, but the states can also be divided differently into Bell local or Bell non-local states. The latter categorisation is based on whether or not the probability $P(a, b|A, B, c)$ for measured outcomes a, b on sub-system observables A, B for state preparation process c , is given by a local hidden variable theory (LHVT) form $P(a, b|A, B, c) = \sum_{\lambda} P(\lambda|c)P(a|A, c, \lambda)P(b|B, c, \lambda)$

(where preparation c results in a probability distribution $P(\lambda|c)$ for hidden variables λ , $P(a|A, c, \lambda)$ is the probability for measured outcome a on sub-system observable A when the hidden variables are λ with $P(b|B, c, \lambda)$ the analogous observable B probability). Quantum states where $P(a, b|A, B, c)$ is given by a LHVT form are Bell local, if not they are Bell non-local and associated with Bell inequality violation experiments. For the Bell local states there are three cases depending on whether both, one of or neither of the LHVT probabilities $P(a|A, c, \lambda)$ and $P(b|B, c, \lambda)$ are also given by a quantum probability involving sub-system density operators. Cases where one or both are given by a quantum probability are known as local hidden states (LHS) and such states are non-steerable. The steerable states are the Bell local states where there is no LHS, or the Bell non-local states.

In a previous paper tests for entanglement for two mode systems involving identical massive bosons were obtained. In the present paper we consider tests for EPR steering and Bell non-locality in such systems. We find that spin squeezing in the any spin component, a weak correlation test, the Hillery-Zubairy spin variance test and a two mode quadrature squeezing test all show that the LHS model fails, and hence the quantum state is steerable. We also find a strong correlation test and a stronger version of the Hillery-Zubairy spin variance test also show that EPR steering occurs. In addition we find a new test for Bell non-locality that applies for measurements based on spin operators.

2 Introduction

Recent papers by Dalton et al [1], [2], [3] have dealt with the topic of bipartite quantum entanglement and experimental tests for demonstrating it in the context of systems of identical massive bosons. However, although the quantum states of composite systems can just be classified into disjoint sets of separable or entangled states, it is also possible to classify them into distinct categories based on local hidden variable theory [4], where the two basic disjoint sub-sets of quantum states are the Bell local states and the Bell non-local states. Based on the work of Wiseman et al [5], [6], [7] the Bell local states can then be divided into three disjoint sub-categories. The four categories of states associated with local hidden variable theory have differing features regarding entanglement, EPR steering and Bell non-locality - as will be explained below. This paper is aimed at developing tests based on experimentally measureable quantities for demonstrating which category applies for quantum states of bipartite composite systems of identical massive bosons.

The local hidden variable theory treatment has origins in papers by Einstein, Schrodinger, Bell and Werner ([8], [9], [10], [4], [11]). Einstein suggested that quantum theory, though correct was incomplete - in that it did not deal satisfactorily with the issue of whether the possible measured outcomes for observable quantities (such as position and momentum) could be regarded as elements of reality irrespective of whether an actual measurement has taken place. This issue was raised in the context of entangled states such as for the EPR paradox, in which the entangled state for two well-separated and no longer interacting sub-systems had well-defined values for the position difference and the momentum sum. Here the choice of measuring the position or momentum for one sub-system would instantaneously determine affect the outcome for measuring position or momentum of the other sub-system - a feature we now refer to as steering - but which Einstein called "spooky action at a distance" and regarded as being in conflict with causality The Schrodinger cat paradox [10] is another example. Bohm [12] described a similar paradox to EPR, but now involving a system consisting of two spin 1/2 particles in a singlet state, and where observables with quantised measured outcomes were now involved. Einstein suggested that quantum theory could be the statistical outcome of a more deterministic underlying theory - essentially what we now refer to as a hidden variable theory - where in its simplest form the possible measured outcomes for all observables always have specific values, and measurement merely revealed what these values are. This is in direct contradiction to the Copenhagen interpretation of quantum theory, in which the values for observables do not have a presence in reality until measurement takes place. However, it was not until 1965 before a quantitative general form for hidden variable theory which could be tested in experiments was proposed by Bell [4]. Basically, the key idea is that hidden variables are determined probabilistically when the state for the composite system is prepared and these would determine the values for all the sub-system observables even after the sub-systems have separated - and even if the observables were incompatible (such as two different spin components). Quantum

states for composite systems that could be described by hidden variable theory (the Bell local states) were such that certain inequalities would apply involving the mean values of products for the results of measuring pairs of observables for both sub-systems - the Bell inequalities [4], [13]. Based on the entangled singlet state of two spin 1/2 particles Clauser et al [14] proposed an experiment that could demonstrate a violation of a Bell inequality. This would show that hidden variable theory could not account for experiments that can be explained by quantum theory. In more elaborate versions of hidden variable theory (see [2] for a description), the hidden variables would merely determine the probabilities of measurement outcomes for each of the sub-systems, whilst the overall expressions for the joint sub-system measurement outcomes are still obtained via classical probability theory. Bell inequalities of the same form still apply in this more elaborate version of hidden variable theory. Subsequent experimental work violating Bell inequalities confirmed that there are some quantum states for which a hidden variable theory does not apply and where quantum theory was needed to explain the results (see Brunner et al [15] for a recent review). States for which a hidden variable theory does not apply (and hence violate Bell inequalities) are the Bell non-local states. It was recognised [11] that all separable states could be described by hidden variable theory (and hence are Bell local) and hence a state had to be entangled to be Bell non-local. However, Werner [11] showed that some entangled states could also be described by hidden variable theory - and hence not violate a Bell inequality. The relationship between the classification of states into separable or entangled on one hand, and a classification into Bell local and Bell non-local states on the other hand is therefore not a simple one. In addition to Bell locality or non-locality, there is the question of which categories of states demonstrate the feature of steering [8], [9], [10], in which a choice of measurement on one sub-system can be used to instantly affect the outcomes for possible measurements on the other sub-system - even if they are well separated. As we will see, steerability requires the absence of the so-called local hidden states - which are sub-system quantum states whose density operator is specified by the hidden variables.

In the work by Wiseman et al, [5], [6], [7] states for bipartite systems defined in terms of local hidden variable theory were first categorised by whether they are Bell local or Bell non-local. Within the states that are Bell local a more detailed categorisation was made based on a hierarchy of non-disjoint sub-sets - firstly by whether they are EPR steerable or not, and then secondly for EPR non-steerable states by whether they are separable or not. In the present paper we find it convenient to categorise the Bell local states into three sub-sets which are disjoint, but which are still related to the hierarchy of non-disjoint sub-sets introduced by Wiseman et al. The disjoint sub-sets of states are defined by whether two, one or none of the sub-system hidden variable probabilities is associated with a local hidden quantum state determined from the hidden variables. Category 1 states involve two hidden states, and this Bell local sub-set is the same as the separable states - which are non-steerable. Category 2 states involve only one hidden state and for this Bell local sub-set the states are entangled but non-steerable. Category 3 states do not involve any hidden

state, and these Bell local states are both entangled and steerable. We will also designate the states that are Bell non-local as Category 4 states, and these states are both entangled and steerable. It is of some interest to devise tests for which specific category a quantum state falls into. The main focus of this paper is on whether the quantum state is steerable - which means showing that it is not in Category 1 or Category 2. In previous work tests have been obtained (see [3] for details of a range of tests found by various authors) for showing that a state is entangled, which therefore rules them out from being in Category 1. Hence we only need to consider tests for showing that the state is also not in Category 2. A second focus of this paper is to devise a test for Bell non-locality that can be applied when the measurable quantities for the two sub-systems have a range of outcomes other than the more limited $+1, -1$ outcomes considered by Clauser et al [14].

We begin with a brief review of measurement probabilities in bipartite systems, focusing both on quantum expressions for joint measurement probabilities and local hidden variable theory expressions. We then consider the detailed description of how the quantum states for composite systems may be categorised. We relate our categories of states to the hierarchy of sub-sets discussed in Refs.[5], [6], [7], [16], and discuss the question of how to interconvert between quantum and local hidden variable theory quantities. The approach used for bipartite systems involves replacing annihilation and creation operators by quadrature amplitudes, including in expressions for spin operators. Important relationships between the mean values for measurements given by quantum theory and by local hidden variable theory are highlighted.

We then consider various tests for quantum entanglement, EPR steering and Bell non-locality, taking into account that the local hidden states must comply with the local particle number super-selection rule, since they must be possible quantum states for the particular sub-system considered on its own. Tests for quantum entanglement (Bloch vector test, spin squeezing in any spin component S_x , S_y or S_z , Hillery et al spin variance test, strong correlation test, a two mode quadrature squeezing test) can also be applied as tests for EPR steering. In addition, as well as a new strong correlation test, a further test for EPR steering involving the sum of the variances for spin operators S_x , S_y and the mean boson number is obtained following an approach in Cavalcanti et al [17] and He et al [18]. This test is a stronger version of the Hillery spin variance test and also involves the mean value for S_z . Finally, a test for Bell non-locality is then found for the situation where sub-system spin operators are measured, generalising the result in Clauser et al [14] to the situation where the measured outcomes are not restricted to ± 1 . The Bell inequality now involves the mean boson numbers N_A, N_B for the sub-systems.

In the Appendices we discuss quantum features that underlying hidden variable theories need to explain, the physical issues behind EPR steering, and some general features of mean values and variances. The Werner states are also described, since in various parameter regimes they provide examples of the four categories of states in the local hidden variable theory model. The idea behind EPR steering is discussed in an Appendix and details for the derivation of an

EPR steering test presented in another. In a further Appendix we examine the possibility of an EPR steering test based on the difference between the variances of the number difference and number sum.

3 Measurement Probabilities in Bipartite Systems

This paper deals with measurements on *bipartite* composite quantum systems, where we have two *distinguishable* sub-systems A and B which are each associated with measureable physical *observables* Ω_A and Ω_B for which possible *outcomes* are denoted α and β . The composite system exists in various quantum *preparation states*, each symbolised by c . Quantum theory has the key feature that such measurements the occurrence of particular outcomes are specified by *probabilities* rather than being *deterministic*, and the basic quantity of interest is the *joint probability* $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ for measurement of *any* pair of sub-system *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c . As the sub-systems are distinct *simultaneous precise measurement* outcomes apply for the pairs of observables Ω_A and Ω_B in both quantum and hidden variable theory (in the latter case the observables are classical variables and not Hermitian operators). The probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ is of course *real* and *positive* and its sum for all outcomes for both Ω_A and Ω_B is equal to unity. The sum of the joint probability over the possible outcomes α for measuring Ω_A defines the *single probability* $P(\beta | \Omega_B, c)$ for measuring Ω_B with outcome β , *irrespective* of the outcome for measuring Ω_A . A similar definition applies for the single probability $P(\alpha | \Omega_A, c)$ for measuring Ω_A with outcome α , irrespective of the outcome for measuring Ω_B . Thus:

$$\sum_{\alpha, \beta} P(\alpha, \beta | \Omega_A, \Omega_B, c) = 1. \quad (1)$$

$$P(\beta | \Omega_B, c) = \sum_{\alpha} P(\alpha, \beta | \Omega_A, \Omega_B, c) \quad (2)$$

$$P(\alpha | \Omega_A, c) = \sum_{\beta} P(\alpha, \beta | \Omega_A, \Omega_B, c) \quad (3)$$

The single probabilities also satisfy the expected *probability sum rules*

$$\sum_{\beta} P(\beta | \Omega_B, c) = 1 \quad \sum_{\alpha} P(\alpha | \Omega_A, c) = 1 \quad (4)$$

which follow from (1).

From the joint measurement probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ and the single measurement probabilities $P(\alpha | \Omega_A, c)$, $P(\beta | \Omega_B, c)$ we can introduce *conditional probabilities* $P(\beta | \Omega_B || \alpha, \Omega_A, c)$ and $P(\alpha | \Omega_A || \beta, \Omega_B, c)$. Here $P(\beta | \Omega_B || \alpha, \Omega_A, c)$ is the probability that measurement of the observable Ω_B yields the outcome β given that measurement of the observable Ω_A yields the outcome α . This (and the corresponding expression for $P(\alpha | \Omega_A || \beta, \Omega_B, c)$) is given by *Bayes' theorem* as

$$\begin{aligned} P(\beta | \Omega_B || \alpha, \Omega_A, c) &= \frac{P(\alpha, \beta | \Omega_A, \Omega_B, c)}{P(\alpha | \Omega_A, c)} \\ P(\alpha | \Omega_A || \beta, \Omega_B, c) &= \frac{P(\alpha, \beta | \Omega_A, \Omega_B, c)}{P(\beta | \Omega_B, c)} \end{aligned} \quad (5)$$

All these expressions apply irrespective of whether the joint and single measurement probabilities are obtained from *quantum theory* or *local hidden variable theory* formulae.

Discussions of *local hidden variable* theory are usually framed in terms of Einstein's viewpoint that although he accepted that quantum theory provided *correct* expressions for joint and single measurement probabilities on composite quantum systems, he felt that such probabilities could *also* be obtained from an underlying *realist* theory based on what are usually referred to as *hidden variables*. Accordingly, we follow this approach and begin by first presenting the *quantum theory expressions* for joint and single measurement probabilities for composite quantum systems, and then the possible underlying hidden variable theory expressions. These quantum expressions (6), (7) and (8) will be regarded as *always* applying - irrespective of additional local hidden variable theory formulae that *may* apply *as well*. For quantum theory, the preparation process is reflected in the *density operator* for the system $c \rightarrow \hat{\rho}(c)$. Only *von Neumann* measurements will be considered. Furthermore, in accordance with the viewpoint that quantum theory is correct, the measurement outcomes α and β will be taken to be the same as the possible outcomes for the corresponding quantum *Hermitian operators* $\hat{\Omega}_A$ and $\hat{\Omega}_B$ - and which in some cases may be *quantized*.

3.1 Quantum Theory - Measurement Probabilities

In *quantum theory* the *joint probability* $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ for measurement of *any* pair of sub-system *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c is given by an expression based on the sub-system observables Ω_A and Ω_B being represented by quantum *Hermitian operators* $\hat{\Omega}_A$ and $\hat{\Omega}_B$. Here simultaneous precise measurement applies because the system operators involved, $\hat{\Omega}_A \otimes \hat{1}_B$ and $\hat{1}_A \otimes \hat{\Omega}_B$ *commute* and therefore have complete sets of simultaneous eigen vectors.

We have for the *joint measurement probability* (see [5], Eq. (2))

$$P(\alpha, \beta | \Omega_A, \Omega_B, c) = \text{Tr}((\hat{\Pi}_\alpha^A \otimes \hat{\Pi}_\beta^B) \hat{\rho}) \quad (6)$$

where $\hat{\Pi}_\alpha^A$ and $\hat{\Pi}_\beta^B$ are *projectors* onto the *eigenvector spaces* for $\hat{\Omega}_A$ and $\hat{\Omega}_B$ associated with the real *eigenvalues* α and β that in quantum theory are the *possible* measurement outcomes. We have $\hat{\Omega}_A \hat{\Pi}_\alpha^A = \alpha \hat{\Pi}_\alpha^A = \hat{\Pi}_\alpha^A \hat{\Omega}_A$, and similar expressions for $\hat{\Pi}_\beta^B$. Clearly the quantum expression for the joint probability satisfies the general probability requirement (1) - the sum rules over α and β being based on the *projector properties* $\sum_\alpha \hat{\Pi}_\alpha^A = \hat{1}^A$ and $\sum_\beta \hat{\Pi}_\beta^B = \hat{1}^B$ involving the sub-system *unit operators* and $\text{Tr} \hat{\rho} = 1$.

The quantum theory expressions for the *single measurement probabilities*

$$\begin{aligned} P(\alpha | \Omega_A, c) &= \text{Tr}((\hat{\Pi}_\alpha^A \otimes \hat{1}^B) \hat{\rho}) \\ P(\beta | \Omega_B, c) &= \text{Tr}((\hat{1}^A \otimes \hat{\Pi}_\beta^B) \hat{\rho}) \end{aligned} \quad (7)$$

for (respectively) measuring Ω_A to have outcome α irrespective of Ω_B and β or for measuring Ω_B to have outcome β irrespective of Ω_A and α both follow from (2) or (3) and the projector properties. The single measurement probabilities can be expressed in terms of *reduced density operators* $\hat{\rho}^A$ and $\hat{\rho}^B$ for the sub-systems

$$\begin{aligned}\hat{\rho}^A &= Tr_B(\hat{\rho}) & \hat{\rho}^B &= Tr_A(\hat{\rho}) \\ P(\alpha|\Omega_A, c) &= Tr_A(\hat{\Pi}_\alpha^A \hat{\rho}^A) & P(\beta|\Omega_B, c) &= Tr_B(\hat{\Pi}_\beta^B \hat{\rho}^B)\end{aligned}\quad (8)$$

The proof of the results (8) for $P(\alpha|\Omega_A, c)$ and $P(\beta|\Omega_B, c)$ is straight-forward. Note that in general the reduced density operators require first knowing the *overall* system density operator $\hat{\rho}$. The joint and single measurement probabilities are related via (3) and (2), as easily shown using $\sum_\alpha \hat{\Pi}_\alpha^A = \hat{1}^A$ and $\sum_\beta \hat{\Pi}_\beta^B = \hat{1}^B$. Using similar considerations and $Tr \hat{\rho} = 1$, the single probabilities also satisfy the sum rules (4).

The *mean value* for joint measurement outcomes of the observables $\hat{\Omega}_A$ and $\hat{\Omega}_B$ will be given by

$$\begin{aligned}\langle \hat{\Omega}_A \otimes \hat{\Omega}_B \rangle &= \sum_{\alpha, \beta} \alpha \beta P(\alpha, \beta|\Omega_A, \Omega_B, c) \\ &= Tr(\hat{\Omega}_A \otimes \hat{\Omega}_B) \hat{\rho}\end{aligned}\quad (9)$$

where the results $\sum_\alpha \alpha \hat{\Pi}_\alpha^A = \hat{\Omega}_A$ and $\sum_\beta \beta \hat{\Pi}_\beta^B = \hat{\Omega}_B$ and (6) have been used.

The mean value for the measurement of a single observable $\hat{\Omega}_A$ is

$$\langle \hat{\Omega}_A \rangle = \sum_\alpha \alpha P(\alpha|\Omega_A, c) = Tr(\hat{\Omega}_A \otimes \hat{1}_B) \hat{\rho} = Tr_A(\hat{\Omega}_A \hat{\rho}^A) \quad (10)$$

as can be derived from (6) and (8).

It is worth noting that for systems of identical massive bosons super-selection rules (SSR) require the overall density operator $\hat{\rho}$ to commute with the *total* number operator N (*global* particle number SSR - see for example Refs.[2], [3] and references therein for discussions on SSR). Consequently the density operator for a two mode system

$$\hat{\rho} = \sum_{n_A, n_B} \sum_{m_A, m_B} \rho(n_A, n_B; m_A, m_B) (|n_A\rangle \otimes |n_B\rangle) (\langle m_A| \otimes \langle m_B|) \quad (11)$$

is such that $\rho(n_A, n_B; m_A, m_B) = 0$ unless $n_A + n_B = m_A + m_B$. It is then straightforward to show that the reduced density operator $\hat{\rho}^A$ for mode A is given by

$$\hat{\rho}^A = \sum_{n_A} (\sum_{n_B} \rho(n_A, n_B; n_A, n_B)) (|n_A\rangle \langle n_A|) \quad (12)$$

which is SSR compliant for the *sub-system* particle number N_A (*local* particle number SSR). This feature will turn out to be relevant for evaluating terms associated with the EPR steering tests.

3.2 Local Hidden Variable Theory - Measurement Probabilities

A *local hidden variable* (LHV) theory is based on hidden variables λ which describe the *real* or *underlying* state of the system, and which are determined with a *probability* $P(\lambda|c)$ for a preparation process c . The probability $P(\lambda|c)$ is real, positive and its sum over all possible hidden variables is also unity. Thus

$$\sum_{\lambda} P(\lambda|c) = 1. \quad (13)$$

The preparation process is thus reflected in the *probability function* for the hidden variables $c \rightarrow P(\lambda|c)$. The *overall picture* is that although the hidden variables are *global* and associated with the original preparation process, they *act* locally in determining the probabilities of *separate* measurements on the *sub-systems* - even in the situation where the sub-systems are localised in *well-separated* spatial regions and the two sub-system measurements occur *simultaneously*.

In *hidden variable theory* the *joint probability* $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for measurement of *any* pair of sub-system *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c is given by an expression based on the sub-system observables Ω_A and Ω_B being represented by *c-numbers*. Sub-system measurement probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ are introduced in LHV theory, and which depend on the hidden variables λ . Here $P(\alpha|\Omega_A, c, \lambda)$ is the probability that measurement of the *observable* Ω_A of sub-system A results in *outcome* α when the *hidden variable* are λ , with a similar definition for $P(\beta|\Omega_B, c, \lambda)$. It is important to note that *each* sub-system observable Ω_C has its *own* single measurement outcome probability $P(\gamma|\Omega_C, c, \lambda)$, not necessarily related to those for a *different* observable for that sub-system. The probability functions for different *powers* of the same observable are of course related.

For a *local HVT* the *joint probability* $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for measurement of *any* pair of sub-system *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c is given by (see [5], Eq. (3), [7], Eq. (15))

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\beta|\Omega_B, c, \lambda) P(\lambda|c) \quad (14)$$

showing that the result is determined from a model based on *hidden variables* λ that are first determined (probabilistically) via the *preparation* process, and which then act *locally* to determine the *sub-system* measurement *probabilities* $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$, which are finally combined in accordance with *classical probability theory* to determine the joint measurement probability. We can also write this as

$$\begin{aligned} P(\alpha, \beta|\Omega_A, \Omega_B, c) &= \sum_{\lambda} P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda) P(\lambda|c) \\ P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda) &= P(\alpha|\Omega_A, c, \lambda) P(\beta|\Omega_B, c, \lambda) \end{aligned} \quad (15)$$

where the last equation expresses the requirement that the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda)$ for a preparation leading to hidden variables λ is given by the *product* of sub-system probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ for that global value of λ . States for which the joint probability is given by the local hidden variable theory Eq. (14) are referred to as *Bell local*. State where this does not apply are the *Bell non-local* states.

The *single measurement* probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ must of course satisfy the general requirements of being *real*, *positive* and such that their *sum* over all possible outcomes is *unity* for each value of the LHV in accordance with the general requirements (4). Thus

$$\sum_{\alpha} P(\alpha|\Omega_A, c, \lambda) = 1 \quad \sum_{\beta} P(\beta|\Omega_B, c, \lambda) = 1 \quad (16)$$

By combining (13) and (16) it is straightforward to show that the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ satisfies the standard probability sum rule (1).

The overall probability $P(\alpha|\Omega_A, c)$ that measurement of the *observable* Ω_A of sub-system A results in *outcome* α when the *preparation* process is c irrespective of the outcome for measurement of the *observable* Ω_B of sub-system B is *defined* as in (3), so it is given by the sum over the possible values λ of the hidden variables of the $P(\alpha|\Omega_A, c, \lambda)$ times the preparation probability $P(\lambda|c)$, with a similar expression for $P(\beta|\Omega_B, c)$. Thus

$$\begin{aligned} P(\alpha|\Omega_A, c) &= \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c) \\ P(\beta|\Omega_B, c) &= \sum_{\lambda} P(\beta|\Omega_B, c, \lambda) P(\lambda|c) \end{aligned} \quad (17)$$

Also, we have using (17) and (15)

$$\begin{aligned} P(\alpha|\Omega_A, c, \lambda) &= \sum_{\beta} P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda) \\ P(\beta|\Omega_B, c, \lambda) &= \sum_{\alpha} P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda) \end{aligned} \quad (18)$$

relating the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda)$ for a preparation leading to hidden variables λ to the corresponding single probabilities for the same λ .

Here we have followed the approach of Refs. [5], [6] (but not the notation) where the sub-system probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ are *not necessarily* given by quantum expressions such as (7) though they *may* be. Later we will introduce a *more specific* notation (subscript Q) to distinguish cases where $P(\alpha|\Omega_A, c, \lambda)$ and/or $P(\beta|\Omega_B, c, \lambda)$ are given by quantum expressions from those where they are not. As we will see, this leads to *three* non-overlapping *categories* of quantum LHV states to distinguish the various possibilities for Bell local states. We will then include the Bell non-local states as a *fourth* category.

We can use (14) to obtain an expression for the *expectation value* of the *joint measurement* of observables Ω_A and Ω_B when the preparation process is c . This

will be given by

$$\begin{aligned}\langle \Omega_A \otimes \Omega_B \rangle &= \sum_{\alpha, \beta} \alpha \beta P(\alpha, \beta | \Omega_A, \Omega_B, c) \\ &= \sum_{\lambda} \langle \Omega_A(c, \lambda) \rangle \langle \Omega_B(c, \lambda) \rangle P(\lambda | c)\end{aligned}\quad (19)$$

where $\langle \Omega_A(c, \lambda) \rangle \equiv \langle \Omega_A(\lambda) \rangle$ is the expectation value of observable Ω_A when the preparation process c leads to hidden variables λ , with $\langle \Omega_B(c, \lambda) \rangle \equiv \langle \Omega_B(\lambda) \rangle$ the corresponding expectation value for observable Ω_B . These are given by

$$\begin{aligned}\langle \Omega_A(c, \lambda) \rangle &= \sum_{\alpha} \alpha P(\alpha | \Omega_A, c, \lambda) \\ \langle \Omega_B(c, \lambda) \rangle &= \sum_{\beta} \beta P(\beta | \Omega_B, c, \lambda)\end{aligned}\quad (20)$$

The mean value for the measurement of a single observable Ω_A is

$$\langle \Omega_A \rangle = \sum_{\alpha} \alpha P(\alpha | \Omega_A, c) = \sum_{\lambda} \langle \Omega_A(c, \lambda) \rangle P(\lambda | c) \quad (21)$$

as can be derived from (17) and (20). A similar result applies for $\langle \Omega_B \rangle$.

In a *non-fuzzy* version of LHVT $\langle \Omega_A(c, \lambda) \rangle = \alpha(c, \lambda)$ and $\langle \Omega_B(c, \lambda) \rangle = \beta(c, \lambda)$, where $\alpha(c, \lambda)$ and $\beta(c, \lambda)$ are *specific* allowed outcomes for measurement of the observables when the preparation process c leads to hidden variables λ . Here the hidden variables λ determine *unique* measurement outcomes $\alpha(c, \lambda)$ and $\beta(c, \lambda)$. These may be quantised. In the non-fuzzy case

$$\langle \Omega_A \otimes \Omega_B \rangle = \sum_{\lambda} \alpha(c, \lambda) \beta(c, \lambda) P(\lambda | c) \quad (22)$$

which is a form originally used for $\langle \Omega_A \otimes \Omega_B \rangle$ in Ref. [4]. Thus in a non-fuzzy version of LHVT the hidden variables *uniquely* specify the measurement outcomes, and it is only because the hidden variables are *not known* that they must be averaged over.

3.3 Links between Quantum and Local Hidden Variable Theory

In accordance with Einstein's basic idea that quantum theory predictions for $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ and $P(\alpha | \Omega_A, c)$, $P(\beta | \Omega_B, c)$ are *correct*, but can be *interpreted* in terms of an underlying *reality* represented by a hidden variable theory, it follows that the *same* joint probability in (14) can *also* be determined from the quantum theory expression (6). Similarly for the single measurement probabilities $P(\alpha | \Omega_A, c)$, $P(\beta | \Omega_B, c)$. The *conditional probabilities* are given by the same general expressions (5) that also apply in the quantum theory case. Again, using (16) and (17) the general relationships (3) and (2) between the joint and single measurement probabilities occur. Also, using (16) and (13) we then see that the general relationship (1) applies in LHV theory.

Thus, quantum theory is not wrong, it is merely *incomplete*. A further discussion of the relationship between quantum theory and an underlying hidden variable theory is presented in Appendix 9.

Equating the LHVT (17) and quantum theory (8) expressions for the single measurement *probability* $P(\alpha|\Omega_A, c)$ we see that

$$P(\alpha|\Omega_A, c) = \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c) = \text{Tr}((\hat{\Pi}_{\alpha}^A \otimes \hat{1}^B) \hat{\rho}) = \text{Tr}_A(\hat{\Pi}_{\alpha}^A \hat{\rho}^A) = P(\alpha|\hat{\Omega}_A, c) \quad (23)$$

which shows that the *hidden variable theory probability* $P(\alpha|\Omega_A, c, \lambda)$ associated with single sub-system A measurements and the *reduced density operator* $\hat{\rho}^A$ for sub-system A are inter-related. A similar result applies for $P(\beta|\Omega_B, c)$. However, this relationship does *not* mean that $P(\alpha|\Omega_A, c, \lambda)$ can always be *determined* from a sub-system density operator which is totally independent of the overall quantum state $\hat{\rho}$ - in general the reduced density operator for each sub-system depends on the *full* density operator $\hat{\rho}$. However, when there is a local hidden state the reduced density operator $\hat{\rho}^A$ may be replaced by the form $\hat{\rho}^A(c, \lambda)$ - which is determined specifically for sub-system A for preparation process c via the hidden variables λ .

Similar considerations apply to the joint measurement *probability* $P(\alpha, \beta|\Omega_A, \Omega_B, c)$. We have

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda) P(\lambda|c) = \text{Tr}((\hat{\Pi}_{\alpha}^A \otimes \hat{\Pi}_{\beta}^B) \hat{\rho}) \quad (24)$$

Since LHVT is required to lead to the same results as quantum theory, we can inter-relate the quantum and LHVT *expectation values* of the joint measurement of observables Ω_A and Ω_B when the preparation process is c . Using (9) and (19) we have

$$\langle \hat{\Omega}_A \otimes \hat{\Omega}_B \rangle = \text{Tr}(\hat{\Omega}_A \otimes \hat{\Omega}_B) \hat{\rho} = \sum_{\lambda} \langle \Omega_A(c, \lambda) \rangle \langle \Omega_B(c, \lambda) \rangle P(\lambda|c) = \langle \Omega_A \otimes \Omega_B \rangle \quad (25)$$

in cases where the LHVT can be applied.

In the case of *expectation values* for a single observable, we have similarly

$$\begin{aligned} \langle \hat{\Omega}_A \rangle &= \langle \hat{\Omega}_A \otimes \hat{1}_B \rangle = \text{Tr}(\hat{\Omega}_A \otimes \hat{1}_B) \hat{\rho} = \text{Tr}_A(\hat{\Omega}_A \hat{\rho}^A) \\ &= \sum_{\lambda} \langle \Omega_A(c, \lambda) \rangle P(\lambda|c) = \langle \Omega_A \otimes 1_B \rangle = \langle \Omega_A \rangle \end{aligned} \quad (26)$$

A similar result applies for $\langle \hat{\Omega}_B \rangle$. These results are all useful for *inter-converting* LHVT and quantum theory expressions, insofar as the LHVT *can* account for the quantum results. As we have already noted, there are *actual* Bell non-local states where the quantum results are *not* accountable via LHVT - either *theoretically* or *experimentally*. So it is only when we are considering *Bell local* states that the above inter-relationships can be applied.

We will also need to consider the mean values for observables which in quantum theory are given by the *sum of products* of sub-system Hermitian operators, where the operators for each sub-system do not necessarily commute -

$[\hat{\Omega}_{A1}, \hat{\Omega}_{A2}] \neq 0$ etc.. Thus for

$$\hat{\Omega} = \hat{\Omega}_{A1} \otimes \hat{\Omega}_{B1} + \hat{\Omega}_{A2} \otimes \hat{\Omega}_{B2} \quad (27)$$

the mean value will be given by

$$\begin{aligned} \langle \hat{\Omega} \rangle &= \langle \hat{\Omega}_{A1} \otimes \hat{\Omega}_{B1} \rangle + \langle \hat{\Omega}_{A2} \otimes \hat{\Omega}_{B2} \rangle \\ &= Tr(\hat{\Omega}_{A1} \otimes \hat{\Omega}_{B1})\hat{\rho} + Tr(\hat{\Omega}_{A2} \otimes \hat{\Omega}_{B2})\hat{\rho} \\ &= \sum_{\alpha_1 \beta_1} \alpha_1 \beta_1 P(\alpha_1, \beta_1 | \Omega_{A1}, \Omega_{B1}, c) + \sum_{\alpha_2 \beta_2} \alpha_2 \beta_2 P(\alpha_2, \beta_2 | \Omega_{A2}, \Omega_{B2}, c) \end{aligned} \quad (28)$$

where

$$P(\alpha_1, \beta_1 | \Omega_{A1}, \Omega_{B1}, c) = Tr(\hat{\Pi}_{\alpha_1} \otimes \hat{\Pi}_{\beta_1})\hat{\rho} \quad P(\alpha_2, \beta_2 | \Omega_{A2}, \Omega_{B2}, c) = Tr(\hat{\Pi}_{\alpha_2} \otimes \hat{\Pi}_{\beta_2})\hat{\rho} \quad (29)$$

In local hidden variable theory the corresponding observable is

$$\Omega = \Omega_{A1} \otimes \Omega_{B1} + \Omega_{A2} \otimes \Omega_{B2} \quad (30)$$

and insofar as local hidden variable theory can be used to describe the state, the mean value of Ω is given by

$$\begin{aligned} \langle \Omega \rangle &= \langle \Omega_{A1} \otimes \Omega_{B1} \rangle + \langle \Omega_{A2} \otimes \Omega_{B2} \rangle \\ &= \sum_{\lambda} P(\lambda | c) \langle \Omega_{A1}(\lambda) \rangle \langle \Omega_{B1}(\lambda) \rangle + \sum_{\lambda} P(\lambda | c) \langle \Omega_{A2}(\lambda) \rangle \langle \Omega_{B2}(\lambda) \rangle \\ &= \sum_{\alpha_1 \beta_1} \alpha_1 \beta_1 P(\alpha_1, \beta_1 | \Omega_{A1}, \Omega_{B1}, c) + \sum_{\alpha_2 \beta_2} \alpha_2 \beta_2 P(\alpha_2, \beta_2 | \Omega_{A2}, \Omega_{B2}, c) \end{aligned} \quad (31)$$

where in LHVT

$$\begin{aligned} P(\alpha_1, \beta_1 | \Omega_{A1}, \Omega_{B1}, c) &= \sum_{\lambda} P(\lambda | c) P(\alpha_1 | \Omega_{A1}, c, \lambda) P(\beta_1 | \Omega_{B1}, c, \lambda) \\ P(\alpha_2, \beta_2 | \Omega_{A2}, \Omega_{B2}, c) &= \sum_{\lambda} P(\lambda | c) P(\alpha_2 | \Omega_{A2}, c, \lambda) P(\beta_2 | \Omega_{B2}, c, \lambda) \end{aligned} \quad (32)$$

We will use these expressions to interconvert between quantum theory and LHVT when the latter applies.

To determine the mean values experimentally, two sets of joint measurements for $\hat{\Omega}_{A1}, \hat{\Omega}_{B1}$ and *then* $\hat{\Omega}_{A2}, \hat{\Omega}_{B2}$ (or the classical observables Ω_{A1}, Ω_{B1} and then Ω_{A2}, Ω_{B2}) would be required, unless a technique exists for measuring the outcomes for $\hat{\Omega}$ (or Ω) directly.

4 Classes of Quantum States for Bipartite Systems

There is various ways the quantum states for bipartite systems can be *categorised* and quantum states falling into a particular category in one scheme *may not* all

end up in the same category in a different scheme. Jones et al [6] (as elaborated by Cavalcanti et al [7]), established a hierarchy of *bipartite quantum states* can be established based on *LHV models* for the *joint probability* $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ for measurement of *any* pair of sub-system *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c . However before considering this hierarchy we first identify a *classification* based purely on *quantum state models*.

4.1 Separable and Entangled States

The quantum states for bipartite composite systems may be divided into *two classes* - the *separable* and the *entangled* states. We will refer to this scheme as the *Quantum Theory Classification Scheme* (QTCS).

The *separable* states are those whose preparation is described by the density operator

$$\hat{\rho}_{sep} = \sum_R P_R \hat{\rho}_R^A \otimes \hat{\rho}_R^B \quad (33)$$

where $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ are *possible* quantum states for sub-systems A and B respectively and P_R is the probability that this *particular pair* of sub-system states is prepared. This follows the preparation process for separable states described by Werner [11]. Such quantum states are of the same form as what Werner [11] referred to as *uncorrelated states*, but which nowadays would be referred to as separable or *non-entangled* states. The *entangled* states are simply the quantum states that are *not* separable.

A detailed discussion of the significance of separable and entangled states, and tests for distinguishing these is given in many articles and textbooks (see for example [2], [3]). Clearly a given quantum state is *either* separable *or* entangled - it cannot be both. Werner [11] referred to the entangled states as *EPR correlated states*, but which nowadays would be referred to as entangled states.

For the present we note that *if* the quantum state is *separable* then from (6) and (33) the joint probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ is given by

$$P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_R P_R \text{Tr}_A(\hat{\Pi}_\alpha^A \hat{\rho}_R^A) \text{Tr}_B(\hat{\Pi}_\beta^B \hat{\rho}_R^B) \quad (34)$$

$$= \sum_R P_R P(\alpha | \Omega_A, c(A, R)) P(\beta | \Omega_B, c(B, R)) \quad (35)$$

where

$$\begin{aligned} P(\alpha | \Omega_A, c(A, R)) &= \text{Tr}_A(\hat{\Pi}_\alpha^A \hat{\rho}_R^A) \\ P(\beta | \Omega_B, c(B, R)) &= \text{Tr}_B(\hat{\Pi}_\beta^B \hat{\rho}_R^B) \end{aligned} \quad (36)$$

are the *sub-system probabilities* for outcomes α, β for measurements of observables Ω_A, Ω_B when the sub-system preparations are $c(A, R) \rightarrow \hat{\rho}_R^A, c(B, R) \rightarrow \hat{\rho}_R^B$. We will return to this result later.

Alternatively, *if* the joint probability is given by (34) for *all* observables and outcomes then

$$P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_R P_R \text{Tr}(\hat{\Pi}_\alpha^A \otimes \hat{\Pi}_\beta^B) (\hat{\rho}_R^A \otimes \hat{\rho}_R^B) = \text{Tr}((\hat{\Pi}_\alpha^A \otimes \hat{\Pi}_\beta^B) \hat{\rho}) \quad (37)$$

where $\hat{\rho} = \sum_R P_R \hat{\rho}_R^A \otimes \hat{\rho}_R^B$ so that the state is separable. Thus the density operator definition and the joint probability expression for a separable state are *equivalent*.

4.2 Bell Local and Non-Local States

The quantum states for bipartite composite systems may *also* be *differently* divided into *two other classes* - the *Bell local* and the *Bell-non-local* states. We will refer to this scheme as the *Local Hidden Variable Theory Classification Scheme* (LHVTCs). As we will see, there is *no simple* relationship between the entangled states on the one hand and the Bell non-local states on the other, (nor between the separable states on the one hand and the Bell local states on the other).

The *Bell local* states are those for which the joint probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ is given by the LHV theory expression (14) *as well as* the quantum theory expression (6). In contrast, the *Bell non-local* states are those for which there is *no* LHV theory expression (14) for the joint probability - this is *only* given by the quantum theory expression (6). As explained in many textbooks and articles (see for example [2]) the Bell local states obey the *Bell inequalities* [4], and the existence of some quantum states (such as the two qubit *Bell states* REFS) for which the Bell inequalities are *not* obeyed or confirmed experimentally shows that Einstein's hope that an underlying reality represented by a local hidden variable theory could underpin quantum theory is *not* realised.

Before looking at a *further classes* of quantum states defined in terms of LHV theory we first present an important result, namely that *all separable* states are *Bell local*. The *formal similarity* between the hidden variable theory expression for the joint probability (14) and the quantum expression (35) for a *separable* state is noticeable. We can identify the probabilistic choice R for the preparation of the *particular pair* of sub-system states $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ with a *particular choice* of hidden variables λ , thus $R \rightarrow \lambda$. Then the probability P_R for this particular pair of sub-system states $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ can be identified with the hidden variable creation probability $P(\lambda | c)$, thus $P_R \rightarrow P(\lambda | c)$. Next, the probabilities $P(\alpha | \Omega_A, c(A, R))$ and $P(\beta | \Omega_B, c(B, R))$ for the single sub-system probabilities will be identified with the hidden variable probabilities $P(\alpha | \Omega_A, c, \lambda)$ and $P(\beta | \Omega_B, c, \lambda)$, thus $P(\alpha | \Omega_A, c(A, R)) \rightarrow P(\alpha | \Omega_A, c, \lambda)$ and $P(\beta | \Omega_B, c, \lambda) \rightarrow P(\beta | \Omega_B, c, \lambda)$. With these identifications the joint probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ for a separable state (35) *is* of the general form for the joint probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ for a Bell local state (14). Hence the separable states are Bell local.

An immediate corollary of this result that all quantum separable states are Bell local is that *all* Bell *non-local* states are *quantum entangled*. After all, if the quantum state is Bell non-local then the LHV theory expression (14) for the joint probability does *not* apply - whereas it *does* apply if the quantum state is separable. Hence Bell non-local states *cannot* be separable and thus *must* be quantum entangled. Thus, these last results therefore provide a general relationship between the classes of quantum states based just on quantum theory and the classes based on local hidden variable theory. This is that *all* quantum separable states are *Bell local* and *all* Bell non-local states are quantum *entangled*. Note however that the converses are *not* true. As we will see, *some* Bell local states are *not* quantum separable, that is they are quantum entangled. Similarly, *some* quantum entangled states are *not* Bell non-local, that is they are Bell local. This last result was established by Werner [11].

4.3 Categories of Bell Local States

In regard to the quantum *separable* states we have from before that the single probabilities are given by *quantum theory* expressions

$$\begin{aligned} P(\alpha|\Omega_A, c, \lambda) &= \text{Tr}_A(\hat{\Pi}_\alpha^A \hat{\rho}_R^A) = P_Q(\alpha|\Omega_A, c, \lambda) \\ P(\beta|\Omega_B, c, \lambda) &= \text{Tr}_B(\hat{\Pi}_\beta^B \hat{\rho}_R^B) = P_Q(\beta|\Omega_B, c, \lambda) \end{aligned} \quad (38)$$

where the subscript Q indicates that a quantum theory expression applies. Hence for *separable states* we have

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_\lambda P_Q(\alpha|\Omega_A, c, \lambda) P_Q(\beta|\Omega_B, c, \lambda) P(\lambda|c) \quad (39)$$

where the single probabilities *are* given by quantum theory expressions.

This particular situation for separable states suggests that the *Bell local states* for *bipartite* systems may be divided up into *three* classes depending on the *number* of single sub-system probabilities that are *definitely* described by quantum expressions. There are three possibilities - (1) *both* $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ are given by quantum expressions as in (38) involving a sub-system *density operator* and a *projector*, (2) only *one* is given by such a quantum expression and (3) *neither* is given by a quantum expression. The three classes or categories are mutually exclusive - a given Bell local state can only be in *one* of the three classes. In the cases where a quantum density operator $\hat{\rho}^{A,B}(c, \lambda)$ *is* involved - which is specified by hidden variables λ , we refer to such a state as a *local hidden state*. We now introduce a *different* notation in which (as in (38)) the *presence* of the sub-script Q on a sub-system LHV probability indicates that it *can* be obtained from a quantum expression involving a sub-system density operator, and the *absence* of the sub-script Q indicates that it *cannot* be obtained from a quantum expression. For cases when the sub-system LHV probability *cannot* be given by a quantum expression involving a sub-system density operator, the sub-system probability does not have a subscript Q and

the probability is given by an upper case notation of the form $P(\alpha|\Omega_A, c, \lambda)$. Note that this notation *differs* from that in Ref [7], wherein the $P(\alpha|\Omega_A, c, \lambda)$ *may* be given by a quantum expression, and from that in Refs. [5], [6] where the $P(\alpha|\Omega_A, c, \lambda)$ could be *either* $P(\alpha|\Omega_A, c, \lambda)$ (non-quantum) *or* $P_Q(\alpha|\Omega_A, c, \lambda)$ (quantum) in our notation. Hence in this notation the joint probabilities for the *Bell local* states in *Categories 1, 2 and 3* are given by

$$\begin{aligned} P(\alpha, \beta|\Omega_A, \Omega_B, c) &= \sum_{\lambda} P_Q(\alpha|\Omega_A, c, \lambda) P_Q(\beta|\Omega_B, c, \lambda) P(\lambda|c) & \text{Category 1} \\ P(\alpha, \beta|\Omega_A, \Omega_B, c) &= \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P_Q(\beta|\Omega_B, c, \lambda) P(\lambda|c) & \text{Category 2} \\ P(\alpha, \beta|\Omega_A, \Omega_B, c) &= \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\beta|\Omega_B, c, \lambda) P(\lambda|c) & \text{Category 3} \end{aligned}$$

As previously indicated, when there is the *absence* of the subscript Q , the single probability is *not* given by a quantum expression. When it is *present*

$$\begin{aligned} P_Q(\alpha|\Omega_A, c, \lambda) &= \text{Tr}_A(\hat{\Pi}_{\alpha}^A \hat{\rho}^A(c, \lambda)) \\ P_Q(\beta|\Omega_B, c, \lambda) &= \text{Tr}_B(\hat{\Pi}_{\beta}^B \hat{\rho}^B(c, \lambda)) \end{aligned} \quad (43)$$

where $\hat{\rho}^A(c, \lambda)$ and $\hat{\rho}^B(c, \lambda)$ are the sub-system density operators for the local hidden states associated with hidden variables λ for preparation c . For definiteness in Category 2 we choose B to be the sub-system where the single probability is given by a quantum expression.

We also list as *Category 4* states those for which the joint probability is *not* given by *any* of Eqs. (40), (41) and (42).

$$\begin{aligned} P(\alpha, \beta|\Omega_A, \Omega_B, c) &\neq \text{Eqs. (40), (41)} \\ &\text{or (42)} & \text{Category 4} \end{aligned} \quad (44)$$

For these states the joint probability is *only* given by the quantum theory expression (6). The Category 4 states are of course the *Bell non-local* states, and as such states *do* occur their existence shows that Einstein's expectation that quantum theory results could be explained via a local hidden variable theory was *not* realised. In Einstein's realist approach there would be *no* Category 4 states.

Clearly, all separable states are Category 1 states, and all Category 1 states are separable, as may be shown by substituting (6) for $P(\alpha, \beta|\Omega_A, \Omega_B, c)$, (38) for the single sub-system probabilities and P_R for $P(\lambda|c)$. The *Category 1* states may also be just referred to as *separable* states. However, *Category 2*, *Category 3* and *Category 4* states must be quantum *entangled* states.

In Refs. [5], [6] and [7] the definition of a *local hidden state (LHS) model* for sub-system B is introduced. These are states in which the sub-system measurement probability $p(\beta|\Omega_B, c, \lambda)$ for sub-system B is given by a quantum expression - $P_Q(\beta|\Omega_B, c, \lambda)$ in our notation, but the sub-system measurement $p(\alpha|\Omega_A, c, \lambda)$ for sub-system A may *either* be given by a quantum expression *or* it may not be. As this is the case for both Category 1 and Category 2 states, we

see that *both* Category 1 and Category 2 states *are* LHS states for sub-system B , since in these cases there is a local hidden state $\hat{\rho}^B(c, \lambda)$ involved. Both Category 3 and Category 4 states are clearly *not* LHS states for sub-system B . In Refs. [5], [6] and [7] the relevant expressions (in *their* notation) that are used to define LHS states for sub-system B are Eqs. (6), (3.6) and (18) respectively.

The feature of *EPR steering* of sub-system B from sub-system A is fully discussed in these three papers, and requires the *failure* of the LHS model for sub-system B . This means that there must be *no* local hidden state $\hat{\rho}^B(c, \lambda)$ for sub-system B . For such states the sub-system B said to be *non-steerable* from sub-system A . For completeness, a brief presentation of the physical argument involved based on a consideration of states that are conditional on the outcomes of measurements on sub-system A , is set out in Appendix 10. Hence Category 1 and Category 2 states are *non-steerable*, whereas Category 3 and Category 4 states are *steerable* since no local hidden state for sub-system B is involved. The Category 3 states, which are Bell local, entangled, non LHS and steerable are sometimes referred to as *EPR entangled* states. Thus, based on their distinction via the number of sub-systems associated with a local hidden state, the four different categories of bipartite states have differing features in regard to *EPR steering*.

As we have now seen, the Bell local states can be divided up into three *non-overlapping* subsets, each of which has different features for the sub-system LHV probabilities $p(\alpha|\Omega_A, c, \lambda)$ and $p(\beta|\Omega_B, c, \lambda)$. This distinctiveness between the sub-sets is of particular convenience when we consider tests for various categories of states. However, it should again be emphasised that other researchers ([5], [6] and [7]) have used a *heirarchy* of non disjoint sub-sets. This is because in certain of their definitions the sub-system probabilities can be either given by quantum or non-quantum expressions. In their scheme the sub-sets overlap, with each set being a sub-set of a larger set. In their scheme Category 1 states (the separable states) would be a sub-set of a set (the LHS states) consisting of Category 1 and Category 2 states. In their scheme the Category 1 and Category 2 states would be combined and be a sub-set of a combined set (the Bell local states) consisting of Category 1, Category 2 and Category 3 states. It is important to note that our scheme and that in Refs.[5], [6] and [7] are *not* the same though they are *related*, and this needs to be taken into account when discussing tests. The *overall scheme* used *here* is shown in Figure 1, where the features for all the different sets of states for bipartite composite systems are set out.

QUANTUM THEORY	LOCAL HIDDEN VAR THEORY	LOCAL HIDDEN VAR THEORY	QUANTUM THY FEATURES	LHVT FEATURES
Separable States	Bell Local States	Category 1	Quantum Separable	LHS State Non-Steer Bell Local
Quantum Entangled States		Category 2	Quantum Entangled	LHS State Non-Steer Bell Local
		Category 3	Quantum Entangled	Steerable Bell Local
		Category 4	Quantum Entangled	Steerable Bell Non-Local

Figure 1. The Quantum Theory and the Local Hidden Variable Theory Classification Schemes (QTCS and LHVCS). The two categories of quantum states in the QTCS are shown in the left column and the two basic categories of quantum states in the LHVCS are shown in the second left column. The four more detailed categories of quantum states in the LHVCS are shown in the third left column, whilst the right two columns lists the features of the four categories of LHVCS states in both the QTCS and LHVCS schemes.

The states introduced by Werner [11] provide examples of the three categories of Bell local states and of the Bell non-local states. These are certain $U \otimes U$ invariant states $((\hat{U} \otimes \hat{U})\hat{\rho}_W(\hat{U}^\dagger \otimes \hat{U}^\dagger) = \hat{\rho}_W$, where \hat{U} is any *unitary* operator) for two d dimensional sub-systems. Depending on the parameter η (or ϕ) the Werner states (see Eq. (145)) , may be separable or entangled. They may also be Bell local and in one of the three categories described above, or they may be Bell non-local. For completeness the Werner states are described in Appendix 11.

5 Spin Squeezing, Bloch Vector, Correlation Tests for EPR Steering

5.1 General Tests for EPR Steering

In a number of papers (see for example MORE REFS, [2], [3]) various tests for standard quantum entanglement have been formulated, recently in the particular context of systems of identical massive bosons [1]. These include spin squeezing, Bloch vector and correlation tests. An important issue then is: Are these tests also valid for detecting *EPR steering* or do some of them fail? Are there new tests for detecting EPR steering? In this situation we are looking for conditions where *LHS model* for sub-system *B* fails - or in other words, the quantum state does *not* have a joint measurement probability as in Eqs. (40) and (41) for Category 1 or Category 2 states. As the tests for quantum entanglement previously obtained have already found the conditions under which Category 1 probabilities fail, we *then* know that the quantum state must be in Category 2, Category 3 or Category 4. If we can then show that it is *not* in Category 2 because the joint measurement probability (41) also fails, then the state *must* be in Category 3 or Category 4 - in other words it is a *EPR steerable* state. We would then have found a *test for EPR steering*. For the Category 2 states the sub-system *A* probabilities $P(\alpha|\Omega_A, c, \lambda)$ in LHVT are *not* given by a quantum expression involving a sub-system density operator. It is *this feature* we must focus on when considering the tests for EPR steering. However, the issue of how to treat mean values and variances in the context of LHV in general and the LHS model in particular requires some consideration, so we have set this out in Appendix 12.

In the present paper, as in previous work [1], [2], [3], we are focusing on tests for bipartite systems involving *identical massive bosons*. Consequently, when quantum states either for the overall system or for a sub-system are involved these must comply with the *symmetrization* principle and *super-selection rules* involving the total boson number for either the overall system or the sub-system. In particular, for Category 2 states (as well as Category 1 states) the local hidden state $\hat{\rho}^B(c, \lambda)$ for the sub-system *B* that is treated quantum mechanically must have zero coherences between Fock states with differing sub-system boson number N_B . The LHS is still a possible quantum state for sub-system *B*. The issue of super-selection rules is discussed fully in [2].

Also, as in these papers both the overall system and the two sub-systems will be specified in terms of *modes* (or single particle states that the particles may occupy) based on a second quantization treatment, rather than in terms of labeled identical *particles* - as might be thought appropriate in a first quantization method. Cases with differing *numbers* of particles are just different *states* of the (multi) modal system, not different systems, as in first quantization.

In addition, since the mean values of various observables are involved in the tests for showing the state is not Category 2, we can use Eqs. (26) and (25) to replace LHVT theory expressions by quantum theory expressions at suitable

stages in the derivations - both when a sub-system B LHS $\hat{\rho}^B(c, \lambda)$ occurs or when we wish to evaluate the mean value of a sub-system A observable Ω_A allowing for all values of the hidden variables λ . However, there will be situations for Category 2 states where we need to consider the mean value of a sub-system A observable Ω_A when the hidden variables have particular values. In this case some general properties of classical probabilities $P(\alpha|\Omega_A, c, \lambda)$ are useful. One is that the mean of the square of a real observable is never less than the square of the mean for the observable.

$$\langle \Omega_A^2(c, \lambda) \rangle \geq (\langle \Omega_A(c, \lambda) \rangle)^2 \quad (45)$$

Another is a Cauchy inequality

$$\sum_{\lambda} C(\lambda) P(\lambda|c) \geq \left(\sum_{\lambda} \sqrt{C(\lambda)} P(\lambda|c) \right)^2. \quad (46)$$

for $C(\lambda) \geq 0$, such as the case $C(\lambda) = \langle \Omega_A^2(c, \lambda) \rangle$. The proof of the first is elementary, the second is proved in [2].

Finally, since LHVT deals with physical quantities that are classical observables we need to express various non-Hermitian quantum mechanical operators that we need to consider - such as mode annihilation and creation operators - in terms of quantum operators that are Hermitian. This will be considered in the next SubSection, where we introduce pairs of *quadrature operators* to replace the annihilation and creation operators for each mode. As we will see, we also need new *auxiliary* Hermitian operators as well, which are sums of products of quadrature operators and these will also be associated with classical observables in the LHVT. All the physical observables that we need to consider have quantum operators that can be written as linear combinations of products $\hat{\Omega}_A \otimes \hat{\Omega}_B$, where both $\hat{\Omega}_A$ and $\hat{\Omega}_B$ are Hermitian - including cases where $\hat{\Omega}_A = \hat{1}_A$ or $\hat{\Omega}_B = \hat{1}_B$. Such products can then be replaced by $\Omega_A \otimes \Omega_B$, where Ω_A and Ω_B are the corresponding classical observables. Using this procedure both quantum and hidden variable theory expressions can be used for the joint measurement probabilities and mean values.

5.2 Quadrature Amplitudes

We now address the issue relevant to discussing spin squeezing tests in the LHS model of how to deal with mean values of sub-system quantities such as mode annihilation or creation operators where the associated quantum operators are non-Hermitian. We can apply the approach where we defined the mean value of complex combinations of observables Ω_1 and Ω_2 via (171), where $\langle (\Omega_1 + i\Omega_2) \rangle = \langle \Omega_1 \rangle + i \langle \Omega_2 \rangle$ and begin by expressing the quantum mode annihilation or creation operators in terms of operators that are Hermitian. These Hermitian components $\hat{\Omega}_1$ and $\hat{\Omega}_2$ can then be taken to represent observable physical quantities Ω_1 and Ω_2 which can then have LHV probabilities to describe measurable outcomes. Mean values associated with the LHS model can then be treated.

In the case of quantum mode annihilation or creation operators the corresponding Hermitian components are the *quadrature operators* for each mode. In quantum theory these are given by the Hermitian operators

$$\begin{aligned}\hat{x}_A &= \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) & \hat{p}_A &= \frac{1}{\sqrt{2}i}(\hat{a} - \hat{a}^\dagger) \\ \hat{x}_B &= \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger) & \hat{p}_B &= \frac{1}{\sqrt{2}i}(\hat{b} - \hat{b}^\dagger)\end{aligned}\quad (47)$$

which have the same commutation rules as the position and momentum operators for distinguishable particles in units where $\hbar = 1$. Thus $[\hat{x}_A, \hat{p}_A] = [\hat{x}_B, \hat{p}_B] = i$ as for cases where A, B were distinguishable particles. By scaling these operators with a length scale given by the Compton wavelength $\lambda_C = \hbar/mc$ and a momentum scale given by the Compton momentum $p_C = mc$, so that with $\hat{X}_{A,B} = \lambda_C \hat{x}_{A,B}$ and $\hat{P}_{A,B} = p_C \hat{p}_{A,B}$ we then have $[\hat{X}_A, \hat{P}_A] = [\hat{X}_B, \hat{P}_B] = i\hbar$, just as for normal quantum position and momentum operators. It is then reasonable to assume that there are equivalent classical observables X_A, P_A, X_B, P_B and that their measurement outcomes would be real numbers, and further more for subsystems not being treated quantum mechanically (such as subsystem A in the context of the LHS model) these outcomes can *actually* be measured in *experiment* and probabilities and mean values such as $P(\alpha|\Omega_A, c, \lambda)$ and $\langle\Omega_A(\lambda)\rangle$ can be assigned as in a hidden variable treatment of subsystem A . In considering Category 2 states the probabilities and mean values such as $P(\beta|\Omega_B, c, \lambda)$ and $\langle\Omega_B(\lambda)\rangle$ for the subsystem B are also given by quantum expressions involving sub-system density operators $\hat{\rho}^B(\lambda)$.

We can write the mode annihilation and creation operators in terms of the quadrature operators via

$$\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2}}(\hat{x}_A + i\hat{p}_A) & \hat{a}^\dagger &= \frac{1}{\sqrt{2}}(\hat{x}_A - i\hat{p}_A) \\ \hat{b} &= \frac{1}{\sqrt{2}}(\hat{x}_B + i\hat{p}_B) & \hat{b}^\dagger &= \frac{1}{\sqrt{2}}(\hat{x}_B - i\hat{p}_B)\end{aligned}\quad (48)$$

which can also be written in terms of the scaled quadrature amplitudes.

We then find that the *spin operators* (defined as $\hat{S}_x = (\hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b})/2$, $\hat{S}_y = (\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b})/2i$, $\hat{S}_z = (\hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{a})/2$) and the *number operators* (defined as $\hat{N} = \hat{N}_A + \hat{N}_B$ with $\hat{N}_A = \hat{a}^\dagger \hat{a}$, $\hat{N}_B = \hat{b}^\dagger \hat{b}$, the separate mode number operators - note that $\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{\hat{N}}{2}(\frac{\hat{N}}{2} + 1)$), can be expressed in terms of the quadrature operators as

$$\begin{aligned}\hat{S}_x &= \frac{1}{2}(\hat{x}_A \hat{x}_B + \hat{p}_A \hat{p}_B) & \hat{S}_y &= \frac{1}{2}(\hat{p}_A \hat{x}_B - \hat{x}_A \hat{p}_B) \\ \hat{S}_z &= \frac{1}{4}(\hat{x}_B^2 - \hat{x}_A^2 + \hat{p}_B^2 - \hat{p}_A^2) - \frac{1}{2}\hat{V}_B + \frac{1}{2}\hat{V}_A \\ \hat{N} &= \frac{1}{2}(\hat{x}_B^2 + \hat{x}_A^2 + \hat{p}_B^2 + \hat{p}_A^2) - \hat{V}_B - \hat{V}_A\end{aligned}\quad (49)$$

which are all linear combinations of products of two quadrature operators. Here we have introduced the *auxiliary* Hermitian operators

$$\begin{aligned}\hat{V}_A &= \frac{1}{2i}(\hat{x}_A\hat{p}_A - \hat{p}_A\hat{x}_A) = \frac{1}{2}\hat{1}_A \\ \hat{V}_B &= \frac{1}{2i}(\hat{x}_B\hat{p}_B - \hat{p}_B\hat{x}_B) = \frac{1}{2}\hat{1}_B\end{aligned}\quad (50)$$

In terms of the quadrature and auxiliary operators the *mode number* and *mode number difference* operators (defined as $\hat{N}_- = \hat{N}_B - \hat{N}_A$) are

$$\hat{N}_A = \frac{1}{2}(\hat{x}_A^2 + \hat{p}_A^2) - \hat{V}_A \quad \hat{N}_B = \frac{1}{2}(\hat{x}_B^2 + \hat{p}_B^2) - \hat{V}_B \quad (51)$$

$$\hat{N}_- = \frac{1}{2}(\hat{x}_B^2 + \hat{p}_B^2 - \hat{x}_A^2 - \hat{p}_A^2) - \hat{V}_B + \hat{V}_A \quad (52)$$

so that $\hat{N}_- = 2\hat{S}_z$ as expected.

We also introduce two *further* distinct auxiliary Hermitian combinations of the quadrature operators for each mode

$$\hat{U}_A = \frac{1}{2}(\hat{x}_A\hat{p}_A + \hat{p}_A\hat{x}_A) \quad \hat{U}_B = \frac{1}{2}(\hat{x}_B\hat{p}_B + \hat{p}_B\hat{x}_B) \quad (53)$$

Using the commutation rules the quantum Hermitian operators \hat{U}_A and \hat{U}_B can be expressed in terms of mode annihilation and creation operators as

$$\hat{U}_A = \frac{1}{2i}((\hat{a})^2 - (\hat{a}^\dagger)^2) \quad \hat{U}_B = \frac{1}{2i}((\hat{b})^2 - (\hat{b}^\dagger)^2) \quad (54)$$

The \hat{U}_A and \hat{U}_B operators appear in the expressions for \hat{S}_x^2 , \hat{S}_y^2 and \hat{S}_z^2 . We find that for \hat{S}_x^2

$$\begin{aligned}\hat{S}_x^2 &= \frac{1}{4}(\hat{x}_A^2\hat{x}_B^2 + \hat{p}_A^2\hat{p}_B^2 + \hat{x}_A\hat{p}_A\hat{x}_B\hat{p}_B + \hat{p}_A\hat{x}_A\hat{p}_B\hat{x}_B) \\ &= \frac{1}{4}(\hat{x}_A^2\hat{x}_B^2 + \hat{p}_A^2\hat{p}_B^2 + (\hat{U}_A + i\hat{V}_A)(\hat{U}_B + i\hat{V}_B) + (\hat{U}_A - i\hat{V}_A)(\hat{U}_B - i\hat{V}_B)) \\ &= \frac{1}{4}(\hat{x}_A^2\hat{x}_B^2 + \hat{p}_A^2\hat{p}_B^2) + \frac{1}{2}(\hat{U}_A\hat{U}_B - \hat{V}_A\hat{V}_B)\end{aligned}\quad (55)$$

In the case of \hat{S}_y^2 we get

$$\begin{aligned}\hat{S}_y^2 &= \frac{1}{4}(\hat{p}_A^2\hat{x}_B^2 + \hat{x}_A^2\hat{p}_B^2 - \hat{p}_A\hat{x}_A\hat{x}_B\hat{p}_B - \hat{x}_A\hat{p}_A\hat{p}_B\hat{x}_B) \\ &= \frac{1}{4}(\hat{p}_A^2\hat{x}_B^2 + \hat{x}_A^2\hat{p}_B^2 - (\hat{U}_A - i\hat{V}_A)(\hat{U}_B + i\hat{V}_B) - (\hat{U}_A + i\hat{V}_A)(\hat{U}_B - i\hat{V}_B)) \\ &= \frac{1}{4}(\hat{p}_A^2\hat{x}_B^2 + \hat{x}_A^2\hat{p}_B^2) - \frac{1}{2}(\hat{U}_A\hat{U}_B + \hat{V}_A\hat{V}_B)\end{aligned}\quad (56)$$

The spin operators thus involve the quadrature operators for both modes. In addition to the spin operators we can also define *two mode quadrature operators*

in terms of the quadrature operators for both modes [3]. These depend on a *phase parameter* θ . There are two sets given by

$$\begin{aligned}\hat{X}_\theta(\pm) &= \frac{1}{2} \left(\hat{a} e^{-i\theta} \pm \hat{b} e^{+i\theta} + \hat{a}^\dagger e^{+i\theta} \pm \hat{b}^\dagger e^{-i\theta} \right) \\ \hat{P}_\theta(\pm) &= \frac{1}{2i} \left(\hat{a} e^{-i\theta} \mp \hat{b} e^{+i\theta} - \hat{a}^\dagger e^{+i\theta} \pm \hat{b}^\dagger e^{-i\theta} \right)\end{aligned}\quad (57)$$

It is easy to see that $\hat{P}_\theta(\pm) = \hat{X}_{\theta+\pi/2}(\pm)$ and that $[\hat{X}_\theta(+), \hat{P}_\theta(+)] = [\hat{X}_\theta(-), \hat{P}_\theta(-)] = i$. The Heisenberg uncertainty principle is given by $\langle \Delta X_\theta^2(\pm) \rangle \langle \Delta P_\theta^2(\pm) \rangle \geq 1/4$ and a state is two mode quadrature *squeezed* if one of $\langle \Delta X_\theta^2(\pm) \rangle$ or $\langle \Delta P_\theta^2(\pm) \rangle$ is less than $1/2$. In Reference [3] we showed that *two mode quadrature squeezing* was a sufficiency test for *entanglement*. We can also write the two mode quadrature operators in terms of the single mode quadrature operators as

$$\begin{aligned}\hat{X}_\theta(\pm) &= \frac{1}{\sqrt{2}} (\hat{x}_A \cos \theta + \hat{p}_A \sin \theta \pm \hat{x}_B \cos \theta \pm \hat{p}_B \sin \theta) \\ \hat{P}_\theta(\pm) &= \frac{1}{\sqrt{2}i} (-\hat{x}_A \sin \theta + \hat{p}_A \cos \theta \mp \hat{x}_B \sin \theta \pm \hat{p}_B \cos \theta)\end{aligned}\quad (58)$$

The square of the two mode quadrature operators $\hat{X}_\theta(\pm)$ are given by

$$\begin{aligned}\hat{X}_\theta(\pm)^2 &= \frac{1}{2} \left\{ \hat{x}_A^2 \cos^2 \theta + \hat{p}_A^2 \sin^2 \theta + 2\hat{U}_A \sin \theta \cos \theta \right\} \\ &+ \frac{1}{2} \left\{ \hat{x}_B^2 \cos^2 \theta + \hat{p}_B^2 \sin^2 \theta + 2\hat{U}_B \sin \theta \cos \theta \right\} \\ &\pm \left\{ \hat{x}_A \hat{x}_B \cos^2 \theta + \hat{p}_A \hat{p}_B \sin^2 \theta + \hat{x}_A \hat{p}_B \sin \theta \cos \theta + \hat{p}_A \hat{x}_B \sin \theta \cos \theta \right\}\end{aligned}\quad (59)$$

The expression for $\hat{P}_\theta(\pm)^2$ can be obtained using $\hat{P}_\theta(\pm) = \hat{X}_{\theta+\pi/2}(\pm)$.

The fundamental quantum Hermitian operators $\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B$ for the two mode system plus the auxiliary Hermitian operators $\hat{U}_A, \hat{V}_A, \hat{U}_B, \hat{V}_B$ all correspond to physical quantities that could be measured, with real eigenvalues as the outcomes. In the local hidden variable theory these quantities correspond to classical observables x_A, p_A, x_B, p_B and U_A, V_A, U_B, V_B , for which single observable hidden variable probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ apply - from which joint probabilities $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ can be obtained via (14). The physical observables involved in the tests such as the spin operators, their squares and the number operators can all be expressed in terms of the quadrature and auxiliary operators as sums of products of the form $\hat{\Omega}_A \otimes \hat{\Omega}_B$. For the local hidden variable theory treatment the corresponding classical observables will be the same as the quantum expressions, but now with the quantum Hermitian operators replaced by the corresponding classical observable. Thus we now have

for the classical spin components S_x , S_y and S_z and the number observable N

$$\begin{aligned} S_x &= \frac{1}{2}(x_A x_B + p_A p_B) & S_y &= \frac{1}{2}(p_A x_B - x_A p_B) \\ S_z &= \frac{1}{4}(x_B^2 - x_A^2 + p_B^2 - p_A^2) - \frac{1}{2}V_B + \frac{1}{2}V_A \\ N &= \frac{1}{2}(x_B^2 + x_A^2 + p_B^2 + p_A^2) - V_B - V_A \end{aligned} \quad (60)$$

Also for the sub-system particle numbers and their difference

$$\begin{aligned} N_A &= \frac{1}{2}(x_A^2 + p_A^2) - V_A & N_B &= \frac{1}{2}(x_B^2 + p_B^2) - V_B \\ N_- &= \frac{1}{2}(x_B^2 + p_B^2 - x_A^2 - p_A^2) - V_B + V_A \end{aligned} \quad (61)$$

The two mode quadrature observables are

$$\begin{aligned} X_\theta(\pm) &= \frac{1}{\sqrt{2}}(x_A \cos \theta + p_A \sin \theta \pm x_B \cos \theta \pm p_B \sin \theta) \\ P_\theta(\pm) &= \frac{1}{\sqrt{2}i}(-x_A \sin \theta + p_A \cos \theta \mp x_B \sin \theta \pm p_B \cos \theta) \end{aligned} \quad (62)$$

The related LHVT expressions for S_x^2 and S_y^2 are obvious. The reverse process for the replacement of the classical observables x_A , x_B , p_A , p_B by \hat{x}_A , \hat{x}_B , \hat{p}_A , \hat{p}_B and U_A , U_B , V_A , V_B by \hat{U}_A , \hat{U}_B , \hat{V}_A , \hat{V}_B requires using (47), (53) and (50) to give the correct quantum Hermitian operators. Carrying out this replacement in the classical spin components S_x , S_y and S_z and the number observable N also gives the correct quantum operators, as also occurs for the squares of these observables as well. Once again we emphasise that we only need single measurement LHVT probabilities $P(\alpha|\Omega_A, c, \lambda)$ with $\Omega_A = x_A, p_A, U_A$ or V_A and $P(\beta|\Omega_B, c, \lambda)$ with $\Omega_B = x_B, p_B, U_B$ or V_B to treat the classical observables such as S_x , S_y and S_z and N or $X_\theta(\pm)$, $P_\theta(\pm)$ via hidden variable theory.

The local hidden variable theory for these new observables is defined by several *independent* single measurement probability functions. For x_A , p_A , U_A and V_A these are $P(\alpha_A|x_A, c, \lambda)$, $P(\beta_A|p_A, c, \lambda)$, $P(\xi_A|U_A, c, \lambda)$ and $P(\eta_A|V_A, c, \lambda)$, with analogous probabilities for x_B , p_B , U_B and V_B .

5.3 Means and Variances for Spin Operators - Category 2 States

5.3.1 Mean Values of Spin Components S_x and S_y - Category 2 States

We now consider the mean value for spin components for the Category 2 states. For example in the case of the spin component S_x

$$\begin{aligned}
\langle S_x \rangle &= \sum_{\lambda} P(\lambda|c) \langle S_x(\lambda) \rangle \\
&= \frac{1}{2} \left(\sum_{\lambda} P(\lambda|c) (\langle x_A(\lambda) \rangle \langle x_B(\lambda) \rangle_Q + \langle p_A(\lambda) \rangle \langle p_B(\lambda) \rangle_Q) \right)
\end{aligned} \tag{63}$$

This expression involves the hidden variable mean values for the (classical) observables x_A and p_A of subsystem A and the local hidden state mean values for the quantum quadrature operators \hat{x}_B and \hat{p}_B . The latter must also correspond to quantum mean values, for a physically realisable quantum state for subsystem B . Thus $\langle x_B(\lambda) \rangle_Q = \text{Tr}(\hat{x}_B \hat{\rho}^B(\lambda))$ and $\langle p_B(\lambda) \rangle_Q = \text{Tr}(\hat{p}_B \hat{\rho}^B(\lambda))$. Since subsystem B is to be treated quantum mechanically then the density operator $\hat{\rho}^B(\lambda)$ would be required to both satisfy the *symmetrisation principle* and be *local particle number SSR* compliant. Hence there is a constraint based on the local hidden state $\hat{\rho}^B(\lambda)$ being a *possible state* for sub-system B that requires the state to be local particle number SSR compliant.

In this case then since both \hat{x}_B and \hat{p}_B are just linear combinations of \hat{b} and \hat{b}^\dagger we have

$$\begin{aligned}
\langle x_B(\lambda) \rangle_Q &= \text{Tr} \frac{1}{\sqrt{2}} (\hat{b} + \hat{b}^\dagger) \hat{\rho}^B(\lambda) = 0 \\
\langle p_B(\lambda) \rangle_Q &= \text{Tr} \frac{1}{\sqrt{2}i} (\hat{b} - \hat{b}^\dagger) \hat{\rho}^B(\lambda) = 0
\end{aligned} \tag{64}$$

$$\langle S_x(\lambda) \rangle = 0 \quad \langle S_y(\lambda) \rangle = 0 \tag{65}$$

and thus

$$\langle S_x \rangle = 0 \quad \langle S_y \rangle = 0 \tag{66}$$

Hence reverting to quantum operators using (25) we have

$$\langle \hat{S}_x \rangle = 0 \quad \langle \hat{S}_y \rangle = 0 \tag{67}$$

These two results are the same as for a quantum separable state. We do not need to know the outcome for $\langle x_A(\lambda) \rangle$ or $\langle p_A(\lambda) \rangle$.

5.3.2 Mean Values of Spin Component S_z and Number N - Category 2 States

For the other spin component S_z we find that for the Category 2 states

$$\begin{aligned}
\langle S_z \rangle &= \frac{1}{4} \sum_{\lambda} P(\lambda|c) (\langle x_B^2(\lambda) \rangle_Q + \langle p_B^2(\lambda) \rangle_Q - \langle x_A^2(\lambda) \rangle - \langle p_A^2(\lambda) \rangle) \\
&\quad - \frac{1}{2} \sum_{\lambda} P(\lambda|c) (\langle V_B(\lambda) \rangle_Q - \langle V_A(\lambda) \rangle)
\end{aligned} \tag{68}$$

Note the presence of terms involving V_B and V_A . As in the quantum separable state case $\langle S_z \rangle$ is not necessarily zero.

For the number observable N we find that for the Category 2 states

$$\begin{aligned} \langle N \rangle &= \frac{1}{2} \sum_{\lambda} P(\lambda|c) (\langle x_B^2(\lambda) \rangle_Q + \langle p_B^2(\lambda) \rangle_Q + \langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle) \\ &\quad - \sum_{\lambda} P(\lambda|c) (\langle V_B(\lambda) \rangle_Q + \langle V_A(\lambda) \rangle) \end{aligned} \quad (69)$$

Note the presence of terms involving V_B and V_A . We will return to these results when we consider the spin squeezing and spin variance tests.

5.3.3 Variances of Spin Components S_x and S_y - Category 2 States

Using (25) and the LHVT expression for S_x^2 obtained from (55) we have for Category 2 states

$$\begin{aligned} \langle S_x^2(\lambda) \rangle &= \frac{1}{4} (\langle x_A^2(\lambda) \rangle \langle x_B^2(\lambda) \rangle_Q + \langle p_A^2(\lambda) \rangle \langle p_B^2(\lambda) \rangle_Q) + \frac{1}{2} (\langle U_A(\lambda) \rangle \langle U_B(\lambda) \rangle_Q - \langle V_A(\lambda) \rangle \langle V_B(\lambda) \rangle_Q) \\ \langle S_y^2(\lambda) \rangle &= \frac{1}{4} (\langle p_A^2(\lambda) \rangle \langle x_B^2(\lambda) \rangle_Q + \langle x_A^2(\lambda) \rangle \langle p_B^2(\lambda) \rangle_Q) - \frac{1}{2} (\langle U_A(\lambda) \rangle \langle U_B(\lambda) \rangle_Q + \langle V_A(\lambda) \rangle \langle V_B(\lambda) \rangle_Q) \end{aligned} \quad (70)$$

and as $\langle S_x(\lambda) \rangle = \langle S_x \rangle = 0$ from (65) we see that $\langle \Delta S_x^2(\lambda) \rangle = \langle S_x^2(\lambda) \rangle$ and $\langle \Delta S_y^2(\lambda) \rangle = \langle S_y^2(\lambda) \rangle$.

Using (170) we then have the inequalities for the Category 2 states

$$\begin{aligned} &\langle \Delta S_x^2 \rangle \\ &\geq \sum_{\lambda} P(\lambda|c) \left(\frac{1}{4} (\langle x_A^2(\lambda) \rangle \langle x_B^2(\lambda) \rangle_Q + \langle p_A^2(\lambda) \rangle \langle p_B^2(\lambda) \rangle_Q) + \frac{1}{2} (\langle U_A(\lambda) \rangle \langle U_B(\lambda) \rangle_Q - \langle V_A(\lambda) \rangle \langle V_B(\lambda) \rangle_Q) \right) \\ &\quad \langle \Delta S_y^2 \rangle \\ &\geq \sum_{\lambda} P(\lambda|c) \left(\frac{1}{4} (\langle p_A^2(\lambda) \rangle \langle x_B^2(\lambda) \rangle_Q + \langle x_A^2(\lambda) \rangle \langle p_B^2(\lambda) \rangle_Q) - \frac{1}{2} (\langle U_A(\lambda) \rangle \langle U_B(\lambda) \rangle_Q + \langle V_A(\lambda) \rangle \langle V_B(\lambda) \rangle_Q) \right) \end{aligned} \quad (71)$$

5.3.4 Evaluation of Expressions Needed - Category 2 States

To consider spin squeezing, spin variance and correlation tests for EPR steering based on the Category 2 states we will need to consider the following additional quantum theory based expressions: $\langle x_B^2(\lambda) \rangle_Q$, $\langle p_B^2(\lambda) \rangle_Q$, $\langle V_B(\lambda) \rangle_Q$, $\langle U_B(\lambda) \rangle_Q$ and the following non-quantum expressions $\langle x_A^2(\lambda) \rangle$, $\langle p_A^2(\lambda) \rangle$, $\langle V_A(\lambda) \rangle$.

Starting with the *quantum* theory expressions

$$\begin{aligned}
\langle x_B^2(\lambda) \rangle_Q &= \frac{1}{2} \text{Tr}(\widehat{b} + \widehat{b}^\dagger)^2 \widehat{\rho}^B(\lambda) \\
&= \frac{1}{2} \left(\text{Tr}((\widehat{b})^2 \widehat{\rho}^B(\lambda)) + \text{Tr}((\widehat{b}^\dagger)^2 \widehat{\rho}^B(\lambda)) + \text{Tr}((\widehat{b}\widehat{b}^\dagger) \widehat{\rho}^B(\lambda)) + \text{Tr}((\widehat{b}^\dagger\widehat{b}) \widehat{\rho}^B(\lambda)) \right) \\
&= \text{Tr}((\widehat{b}^\dagger\widehat{b}) \widehat{\rho}^B(\lambda)) + \frac{1}{2} \\
&= \langle N_B(\lambda) \rangle_Q + \frac{1}{2}
\end{aligned} \tag{72}$$

where the commutation rules have been used and the SSR constraints eliminate the $\text{Tr}((\widehat{b})^2 \widehat{\rho}^B(\lambda))$ and $\text{Tr}((\widehat{b}^\dagger)^2 \widehat{\rho}^B(\lambda))$ terms. Similarly

$$\langle p_B^2(\lambda) \rangle_Q = \langle N_B(\lambda) \rangle_Q + \frac{1}{2} \tag{73}$$

Note that $\langle N_B(\lambda) \rangle_Q \geq 0$.

Then using (54) we find that

$$\begin{aligned}
\langle U_B(\lambda) \rangle_Q &= \frac{1}{2i} \text{Tr}((\widehat{b})^2 - (\widehat{b}^\dagger)^2) \widehat{\rho}^B(\lambda) \\
&= 0
\end{aligned} \tag{74}$$

again due to the SSR constraints on the hidden state $\widehat{\rho}^B(\lambda)$.

Also, using (50)

$$\begin{aligned}
\langle V_B(\lambda) \rangle_Q &= \frac{1}{2} \text{Tr}_B(\widehat{1}_B \widehat{\rho}^B(\lambda)) \\
&= \frac{1}{2}
\end{aligned} \tag{75}$$

since the trace of a density operator is unity.

Finally, using (72), (73) and (75) confirm the result that

$$\langle N_B(\lambda) \rangle_Q = \frac{1}{2} \langle x_B^2(\lambda) \rangle_Q + \frac{1}{2} \langle p_B^2(\lambda) \rangle_Q - \langle V_B(\lambda) \rangle_Q \tag{76}$$

from (61).

For the *local hidden variable theory* expressions using (61) we have

$$\langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle = 2 \langle N_A(\lambda) \rangle + 2 \langle V_A(\lambda) \rangle \tag{77}$$

Using the results (72), (73), (74) and (75) and (77) we now have for Category

2 states

$$\begin{aligned}
& \langle \Delta S_x^2 \rangle \\
& \geq \sum_{\lambda} P(\lambda|c) \left(\frac{1}{2} (\langle N_B(\lambda) \rangle_Q + \frac{1}{2}) (\langle N_A(\lambda) \rangle + \langle V_A(\lambda) \rangle) - \frac{1}{4} \langle V_A(\lambda) \rangle \right) \\
& \geq \sum_{\lambda} P(\lambda|c) \left(\frac{1}{2} \langle N_B(\lambda) \rangle_Q \langle N_A(\lambda) \rangle + \frac{1}{2} \langle N_B(\lambda) \rangle_Q \langle V_A(\lambda) \rangle + \frac{1}{4} \langle N_A(\lambda) \rangle \right) \\
& \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{V}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle \\
& \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle \\
& \quad \langle \Delta S_y^2 \rangle \\
& \geq \sum_{\lambda} P(\lambda|c) \left(\frac{1}{2} (\langle N_B(\lambda) \rangle_Q + \frac{1}{2}) (\langle N_A(\lambda) \rangle + \langle V_A(\lambda) \rangle) - \frac{1}{4} \langle V_A(\lambda) \rangle \right) \\
& \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle \tag{78}
\end{aligned}$$

where we have used (50) and the LHVT and quantum theory interconversions (25) and (26) for the Bell local Category 2 states. These inequalities are the same as those for Category 1 states (see [3]). Also from (68)

$$\begin{aligned}
\langle S_z \rangle &= \frac{1}{4} \sum_{\lambda} P(\lambda|c) \left(2 \langle N_B(\lambda) \rangle_Q + 1 - 2 \langle N_A(\lambda) \rangle - 2 \langle V_A(\lambda) \rangle \right) - \frac{1}{2} \sum_{\lambda} P(\lambda|c) \left(\frac{1}{2} - \langle V_A(\lambda) \rangle \right) \\
&= \frac{1}{2} \sum_{\lambda} P(\lambda|c) \left(\langle N_B(\lambda) \rangle_Q - \langle N_A(\lambda) \rangle \right) \\
&= \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle \\
\frac{1}{2} |\langle S_z \rangle| &\leq \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle \tag{79}
\end{aligned}$$

The last line follows from both $\langle \hat{1}_A \otimes \hat{N}_B \rangle = \sum_{n_A n_B} n_B \rho_{n_A n_B; n_A n_B}$ and $\langle \hat{N}_A \otimes \hat{1}_B \rangle = \sum_{n_A n_B} n_A \rho_{n_A n_B; n_A n_B}$ never being negative. This result is the same as that for Category 1 states (see [3]).

Finally, from (69)

$$\begin{aligned}
\langle N \rangle &= \sum_{\lambda} P(\lambda|c) \left(\langle N_B(\lambda) \rangle_Q + \frac{1}{2} + \langle N_A(\lambda) \rangle + \langle V_A(\lambda) \rangle \right) - \sum_{\lambda} P(\lambda|c) \left(\frac{1}{2} + \langle V_A(\lambda) \rangle \right) \\
&= \sum_{\lambda} P(\lambda|c) \left(\langle N_B(\lambda) \rangle_Q + \langle N_A(\lambda) \rangle \right) \\
&= \langle \hat{1}_A \otimes \hat{N}_B \rangle + \langle \hat{N}_A \otimes \hat{1}_B \rangle \tag{80}
\end{aligned}$$

This result is the same as that for Category 1 states (see [3]).

5.4 Means and Variances for Two Mode Quadrature Operators - Category 2 States

5.4.1 Mean Values for Two Mode Quadratures $X_\theta(\pm)$ and $P_\theta(\pm)$ - Category 2 States

We now consider the mean value for two mode quadrature observables for the Category 2 states. For example in the case of the quadratures $X_\theta(\pm)$

$$\langle X_\theta(\pm) \rangle = \frac{1}{\sqrt{2}} \left(\sum_{\lambda} P(\lambda|c) (\langle x_A(\lambda) \rangle \cos \theta + \langle p_A(\lambda) \rangle \sin \theta \pm \langle x_B(\lambda) \rangle_Q \cos \theta \pm \langle p_B(\lambda) \rangle_Q \sin \theta) \right) \quad (81)$$

using Eq.(62). A similar result is found for $P_\theta(\pm)$. We then use the previous results (64) for sub-system B to find

$$\begin{aligned} \langle X_\theta(\pm) \rangle &= \frac{1}{\sqrt{2}} \left(\sum_{\lambda} P(\lambda|c) (\langle x_A(\lambda) \rangle \cos \theta + \langle p_A(\lambda) \rangle \sin \theta) \right) \\ \langle P_\theta(\pm) \rangle &= \frac{1}{\sqrt{2}i} \left(\sum_{\lambda} P(\lambda|c) (-\langle x_A(\lambda) \rangle \sin \theta + \langle p_A(\lambda) \rangle \cos \theta) \right) \end{aligned} \quad (82)$$

5.4.2 Mean Values for Squares of Two Mode Quadratures - Category Two States

Using (25) and the LHVT expression for $X_\theta(\pm)^2$ obtained from (59) we have for Category 2 states,

$$\begin{aligned} \langle X_\theta(\pm)^2 \rangle &= \frac{1}{2} \sum_{\lambda} P(\lambda|c) (\langle x_A^2(\lambda) \rangle \cos^2 \theta + \langle U_A(\lambda) \rangle 2 \sin \theta \cos \theta + \langle p_A^2(\lambda) \rangle \sin^2 \theta) \\ &\quad + \frac{1}{2} \sum_{\lambda} P(\lambda|c) \left(\langle N_B(\lambda) \rangle_Q + \frac{1}{2} \right) \end{aligned} \quad (83)$$

where we have used the previous results (64) and (74) for sub-system B to eliminate terms involving $\langle x_B(\lambda) \rangle_Q$, $\langle p_B(\lambda) \rangle_Q$ and $\langle U_B(\lambda) \rangle_Q$ and the results (72) and (73) for $\langle x_B^2(\lambda) \rangle_Q$ and $\langle p_B^2(\lambda) \rangle_Q$ to simplify the last term.

5.4.3 Evaluation of Variances for Two Mode Quadratures - Category 2 States

We next use the LHVT - quantum theory equivalences (26) to replace (82) and (83) by their quantum forms. Hence for the mean values of the two mode

quadratures we have

$$\langle X_\theta(\pm) \rangle = \langle \hat{X}_\theta(\pm) \rangle = \frac{1}{\sqrt{2}} (\langle \hat{x}_A \rangle \cos \theta + \langle \hat{p}_A \rangle \sin \theta) \quad (84)$$

$$\begin{aligned} \langle X_\theta(\pm)^2 \rangle &= \langle \hat{X}_\theta(\pm)^2 \rangle \\ &= \frac{1}{2} \left(\langle \hat{x}_A^2 \rangle \cos^2 \theta + \langle \hat{U}_A \rangle 2 \sin \theta \cos \theta + \langle \hat{p}_A^2 \rangle \sin^2 \theta \right) \\ &\quad + \frac{1}{2} \left(\langle \hat{N}_B \rangle + \frac{1}{2} \right) \\ &= \frac{1}{2} \left(\langle \hat{x}_A^2 \rangle \cos^2 \theta + \langle (\hat{x}_A \hat{p}_A + \hat{p}_A \hat{x}_A) \rangle \sin \theta \cos \theta + \langle \hat{p}_A^2 \rangle \sin^2 \theta \right) \\ &\quad + \frac{1}{2} \left(\langle \hat{N}_B \rangle + \frac{1}{2} \right) \end{aligned} \quad (85)$$

on substituting for \hat{U}_A from (53).

The variance is then given by

$$\begin{aligned} \langle \Delta X_\theta(\pm)^2 \rangle &= \langle \Delta \hat{X}_\theta(\pm)^2 \rangle \\ &= \frac{1}{2} \left(\left\{ \langle \hat{x}_A^2 \rangle - \langle \hat{x}_A \rangle^2 \right\} \cos^2 \theta + \left\{ \langle \hat{x}_A \hat{p}_A + \hat{p}_A \hat{x}_A \rangle - 2 \langle \hat{x}_A \rangle \langle \hat{p}_A \rangle \right\} \sin \theta \cos \theta + \left\{ \langle \hat{p}_A^2 \rangle - \langle \hat{p}_A \rangle^2 \right\} \sin^2 \theta \right) \\ &\quad + \frac{1}{2} \left(\langle \hat{N}_B \rangle + \frac{1}{2} \right) \\ &= \frac{1}{2} \langle (\Delta \hat{x}_A \cos \theta + \Delta \hat{p}_A \sin \theta) (\Delta \hat{x}_A \cos \theta + \Delta \hat{p}_A \sin \theta) \rangle + \frac{1}{2} \left(\langle \hat{N}_B \rangle + \frac{1}{2} \right) \end{aligned}$$

where $\Delta \hat{x}_A = \hat{x}_A - \langle \hat{x}_A \rangle$, $\Delta \hat{p}_A = \hat{p}_A - \langle \hat{p}_A \rangle$. The expression for $\langle \Delta P_\theta(\pm)^2 \rangle$ can be obtained using $\hat{P}_\theta(\pm) = \hat{X}_{\theta+\pi/2}(\pm)$.

However, we can make use of the SSR to simplify these expressions further. As shown in SubSection 3.1 the reduced density operator for sub-system A satisfies the local particle number SSR. Consequently

$$\langle \hat{x}_A \rangle = \text{Tr}_A(\hat{x}_A \hat{\rho}^A) = \langle x_A \rangle = 0 \quad \langle \hat{p}_A \rangle = \text{Tr}_A(\hat{p}_A \hat{\rho}^A) = \langle p_A \rangle = 0 \quad (87)$$

using the same arguments as for $\langle x_B(\lambda) \rangle_Q$ and $\langle p_B(\lambda) \rangle_Q$ in Eq.(64). Furthermore, the same steps as for $\langle x_B^2(\lambda) \rangle_Q$, $\langle p_B^2(\lambda) \rangle_Q$ and $\langle U_B(\lambda) \rangle_Q$ lead to

$$\begin{aligned} \langle \hat{x}_A^2 \rangle &= \langle x_A^2 \rangle = \langle \hat{N}_A \rangle + \frac{1}{2} & \langle \hat{p}_A^2 \rangle &= \langle p_A^2 \rangle = \langle \hat{N}_A \rangle + \frac{1}{2} \\ \langle \hat{U}_A \rangle &= \langle U_A \rangle = 0 \end{aligned} \quad (88)$$

(see SubSubSection 5.3.4). Using these results we then find that

$$\begin{aligned}
\langle \Delta X_\theta(\pm)^2 \rangle &= \langle \Delta \hat{X}_\theta(\pm)^2 \rangle \\
&= \frac{1}{2} \left(\langle \hat{N}_A \rangle + \frac{1}{2} \right) + \frac{1}{2} \left(\langle \hat{N}_B \rangle + \frac{1}{2} \right) \\
&= \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2} \\
\langle \Delta P_\theta(\pm)^2 \rangle &= \langle \Delta \hat{P}_\theta(\pm)^2 \rangle \\
&= \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2}
\end{aligned} \tag{89}$$

5.5 Tests for EPR Steering - Previous Entanglement Tests ?

Here we consider whether tests that have been shown to be sufficient to demonstrate quantum entanglement (violation of Category 1) (see Ref. [3] for details) are also valid for demonstrating EPR steering. We first consider the Bloch vector (or weak correlation) test, then a strong correlation test, followed by spin squeezing tests for S_z and for other spin components, as well as the Hillery spin variance test. We also consider a test involving the difference of the variances for the number sum and number difference. Of these possible tests, only the Bloch vector and spin squeezing in S_z are valid for demonstrating EPR steering.

5.5.1 Bloch Vector Test - Category 2 States

From (67) for a LHS state we immediately see that if

$$\langle S_x \rangle \neq 0 \quad \text{or} \quad \langle S_y \rangle \neq 0 \tag{90}$$

then the quantum state cannot be in Category 2 (or Category 1) hence reverting to quantum operators the *Bloch vector test* $\langle \hat{S}_x \rangle \neq 0$ or $\langle \hat{S}_y \rangle \neq 0$ now also shows that the state is *EPR steered* since the quantum state must be either Category 3 or Category 4.

5.5.2 Correlation Test - Category 2 States

The quantum operator $\hat{a}^\dagger \hat{b}$ is not an observable, but from the definitions for the spin operator we can write $\hat{a}^\dagger \hat{b} = \hat{S}_x - i\hat{S}_y$. We can then interpret $\hat{a}^\dagger \hat{b}$ to be $S_x - iS_y$, where now S_x and S_y are observables whose mean values are definable in a LHV theory.

From (171) and (67) we see that in the LHS model

$$\begin{aligned}\langle a^\dagger b \rangle &= \langle S_x \rangle - i \langle S_y \rangle \\ &= 0\end{aligned}\tag{91}$$

so that

$$\begin{aligned}|\langle a^\dagger b \rangle|^2 &= \langle S_x \rangle^2 + \langle S_y \rangle^2 \\ &= 0\end{aligned}\tag{92}$$

for quantum states in Category 2 (or Category 1). Hillery and Zubairy [19] showed that for separable states (Category 1 states) that $|\langle \hat{a}^\dagger \hat{b} \rangle|^2 \leq \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle$. The proof of this result was valid irrespective of whether the sub-system states $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ were local particle number SSR compliant or not (see Ref. [3] for details). Hence as the numbers of bosons N_A and N_B are observables in any LHV (and therefore the mean $\langle N_A \otimes N_B \rangle$ can be defined) we see that for Category 1 states

$$|\langle a^\dagger b \rangle|^2 \leq \langle N_A \otimes N_B \rangle\tag{93}$$

So reverting to quantum operators using (25)

$$\begin{aligned}|\langle \hat{a}^\dagger \hat{b} \rangle|^2 &= \langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2 \\ &> \langle \hat{N}_A \otimes \hat{N}_B \rangle\end{aligned}\tag{94}$$

is a possible test for quantum entanglement. The question is whether it is also a possible test for EPR steering, since $\langle \hat{S}_x \rangle = \langle \hat{S}_y \rangle = 0$ for both Category 1 and Category 2 states that are local particle number SSR compliant. However, (93) is trivially true so if $\langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2$ is greater than *any* positive quantity, it would follow that the quantum state is not Category 1 or Category 2. So no useful test for either quantum entanglement or EPR steering involving $\langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2$ and $\langle \hat{N}_A \otimes \hat{N}_B \rangle$ is established at this point. Later however (see Section 5.6) it will be shown that related tests can be obtained both for quantum entanglement and EPR steering - though not for Bell non-locality.

5.5.3 Spin Squeezing S_z Test - Category 2 States

We then immediately see that if the observable S_z is squeezed with respect to S_x or S_z is squeezed with respect to S_y or vice versa, then the LHS model fails, because such spin squeezing requires $\langle \Delta S_z^2 \rangle$ to be less than either $|\langle S_x \rangle|/2$ or $|\langle S_y \rangle|/2$ and this is impossible for a LHS model - where $\langle S_x \rangle = \langle S_y \rangle = 0$ for both Category 1 and Category 2 states. Thus spin squeezing in the observable S_z shows the state is *EPR steered*.

5.5.4 Spin Squeezing in S_x vrs S_y Test - Category 2 States ?

Combining (78) and (79) we find that

$$(\langle \Delta S_x^2 \rangle - \frac{1}{2} |\langle S_z \rangle|) \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle \quad (95)$$

Similarly

$$(\langle \Delta S_y^2 \rangle - \frac{1}{2} |\langle S_z \rangle|) \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle \quad (96)$$

The right side $\frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle = \frac{1}{2} \sum_{n_A n_B} n_A n_B \rho_{n_A n_B ; n_A n_B}$ is never negative.

These result are the same as those for Category 1 states (see [3]). Hence we find that for Category 2 states there is no spin squeezing in S_x compared to S_y (or vice versa). Thus spin squeezing in S_x compared to S_y (or vice versa) is a sufficiency test for showing that the state cannot be in Category 2. As it has already been shown that such spin squeezing is also a sufficiency test for the state not being in Category 1, we see that spin squeezing in *any* spin component is a sufficiency test for *EPR steering*.

5.5.5 Hillary Spin Variance Test - Category 2 States ?

The Hillery spin variance test [19] for quantum entanglement is $\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle < 0$. We now consider the quantity $\langle \Delta S_x^2 \rangle + \langle \Delta S_y^2 \rangle - \frac{1}{2} \langle N \rangle$ for Category 2 states using (78) and (80). We find that (the $\langle V_A(\lambda) \rangle$ cancel)

$$\begin{aligned} & \langle \Delta S_x^2 \rangle + \langle \Delta S_y^2 \rangle - \frac{1}{2} \langle N \rangle \\ & \geq \langle \hat{N}_A \otimes \hat{N}_B \rangle \geq 0 \end{aligned} \quad (97)$$

This result is also valid for Category 1 states (see Ref. [3] for details).

Thus we can say that if

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle < 0 \quad (98)$$

then the Category 2 model fails. It also shows that Category 1 (separable states) fails, this being the Hillery spin variance test [19] for entanglement.

Hence there is a Hillary spin variance inequality test for *EPR steering*.

5.6 Tests for EPR Steering - Strong Correlation and He Spin Variance Test ?

Here we follow the approach set out in Cavalcanti et al (Ref. [17]) and [20] to derive an inequality for $|\langle \hat{a}^\dagger \hat{b} \rangle|^2$ for Category 1, Category 2 and Category 3

states. For Category 1 states the result gives the strong correlation test and the Hillery et al [19] test for quantum entanglement, whilst for Category 2 states the result gives a condition for EPR steering set out in He et al [18] for the case where $\langle \hat{S}_z \rangle = 0$, plus a new test when this is not the case. For Category 3 states no useful test for Bell non-locality occurs.

5.6.1 General Inequality for $|\langle \hat{a}^\dagger \hat{b} \rangle|^2$ - LHVT

We have using (48) to introduce quadrature operators

$$\langle \hat{a}^\dagger \hat{b} \rangle = \frac{1}{2} (\langle \hat{x}_A \hat{x}_B \rangle + \langle \hat{p}_A \hat{p}_B \rangle + i (\langle \hat{x}_A \hat{p}_B \rangle - \langle \hat{p}_A \hat{x}_B \rangle)) \quad (99)$$

so introducing LHVT expressions

$$\langle \hat{a}^\dagger \hat{b} \rangle = \frac{1}{2} \sum_{\lambda} P(\lambda|c) (\langle x_A(\lambda) \rangle - i \langle p_A(\lambda) \rangle) (\langle x_B(\lambda) \rangle + i \langle p_B(\lambda) \rangle) \quad (100)$$

and then as

$$|\langle \hat{a}^\dagger \hat{b} \rangle| \leq \frac{1}{2} \sum_{\lambda} P(\lambda|c) |(\langle x_A(\lambda) \rangle - i \langle p_A(\lambda) \rangle)| |(\langle x_B(\lambda) \rangle + i \langle p_B(\lambda) \rangle)| \quad (101)$$

and $|(\langle x_A(\lambda) \rangle - i \langle p_A(\lambda) \rangle)| = \sqrt{\langle x_A(\lambda) \rangle^2 + \langle p_A(\lambda) \rangle^2}$ etc, we then find that

$$|\langle \hat{a}^\dagger \hat{b} \rangle|^2 \leq \frac{1}{4} \left(\sum_{\lambda} P(\lambda|c) \sqrt{\langle x_A(\lambda) \rangle^2 + \langle p_A(\lambda) \rangle^2} \sqrt{\langle x_B(\lambda) \rangle^2 + \langle p_B(\lambda) \rangle^2} \right)^2 \quad (102)$$

Using the inequality $\sum_R P_R C_R \geq \left(\sum_R P_R \sqrt{C_R} \right)^2$ with $\sum_R P_R = 1$ and $C_R \geq 0$ we then have the key inequality

$$|\langle \hat{a}^\dagger \hat{b} \rangle|^2 \leq \frac{1}{4} \sum_{\lambda} P(\lambda|c) \left(\langle x_A(\lambda) \rangle^2 + \langle p_A(\lambda) \rangle^2 \right) \left(\langle x_B(\lambda) \rangle^2 + \langle p_B(\lambda) \rangle^2 \right) \quad (103)$$

that would follow from the approach in Ref. [17]. In terms of quantum operators we can use (51), (25) and (26) to write this inequality for all states described via LHVT as

$$\begin{aligned} |\langle \hat{a}^\dagger \hat{b} \rangle|^2 &\leq \langle (\hat{N}_A + \hat{V}_A) \otimes (\hat{N}_B + \hat{V}_B) \rangle \\ &= \left\langle \left(\hat{N}_A + \frac{1}{2} \hat{1}_A \right) \otimes \left(\hat{N}_B + \frac{1}{2} \hat{1}_B \right) \right\rangle \\ &= \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle + \frac{1}{4} \end{aligned} \quad (104)$$

so if $|\langle \hat{a}^\dagger \hat{b} \rangle|^2 > \langle (\hat{N}_A + \frac{1}{2} \hat{1}_A) \otimes (\hat{N}_B + \frac{1}{2} \hat{1}_B) \rangle$ then the quantum state cannot be described via a LHVT. However, as we will see this last inequality can be made less difficult to violate for Category 1 and Category 2 states.

5.6.2 Strong Inequalities for Categories 1, 2 and 3 States

Stronger inequalities can now be derived for the quantities $\left(\langle x_A(\lambda) \rangle^2 + \langle p_A(\lambda) \rangle^2\right)$ and $\left(\langle x_B(\lambda) \rangle^2 + \langle p_B(\lambda) \rangle^2\right)$ in the cases of Categories 1, 2 and 3 states. This leads to some outcomes different to (104).

If the sub-system C does *not* involve a local hidden state $\hat{\rho}_\lambda^C$ then we can use Schwarz' inequality to give $\langle x_C(\lambda) \rangle^2 \leq \langle x_C^2(\lambda) \rangle$ and $\langle p_C(\lambda) \rangle^2 \leq \langle p_C^2(\lambda) \rangle$. This is equivalent to the variances of x_C and p_C being non-negative. So $\langle x_C(\lambda) \rangle^2 + \langle p_C(\lambda) \rangle^2 \leq \langle x_C^2(\lambda) \rangle + \langle p_C^2(\lambda) \rangle$.

On the other hand, if the sub-system C *does* involve a local hidden state $\hat{\rho}_\lambda^C$ then we can obtain a *stronger inequality* via quantum theory. For any real η the quantity $\langle (\Delta \hat{x}_C - i\eta \Delta \hat{p}_C) (\Delta \hat{x}_C + i\eta \Delta \hat{p}_C) \rangle_\lambda = \text{Tr} (\Delta \hat{x}_C - i\eta \Delta \hat{p}_C) (\Delta \hat{x}_C + i\eta \Delta \hat{p}_C) \hat{\rho}_\lambda^C \geq 0$, where $\Delta \hat{x}_C = \hat{x}_C - \langle \hat{x}_C \rangle_\lambda$, $\Delta \hat{p}_C = \hat{p}_C - \langle \hat{p}_C \rangle_\lambda$. Thus for all η we have $\langle \Delta \hat{x}_C^2 \rangle_\lambda - \eta + \eta^2 \langle \Delta \hat{p}_C^2 \rangle_\lambda \geq 0$ using $[\hat{x}_C, \hat{p}_C] = i$. Putting $\eta = 1$ gives the inequality $\langle \Delta \hat{x}_C^2 \rangle_\lambda + \langle \Delta \hat{p}_C^2 \rangle_\lambda - 1 \geq 0$, which can be written as $\langle \hat{x}_C \rangle_\lambda^2 + \langle \hat{p}_C \rangle_\lambda^2 \leq \langle \hat{x}_C^2 \rangle_\lambda + \langle \hat{p}_C^2 \rangle_\lambda - 1$. In terms of LHVT notation this inequality is $\langle x_C(\lambda) \rangle^2 + \langle p_C(\lambda) \rangle^2 \leq \langle x_C^2(\lambda) \rangle + \langle p_C^2(\lambda) \rangle - 1$.

For Category 1 states both sub-systems involve a local hidden state, so the key inequality (103) gives

$$|\langle \hat{a}^\dagger \hat{b} \rangle|^2 \leq \frac{1}{4} \sum_\lambda P(\lambda|c) (\langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle - 1) (\langle x_B^2(\lambda) \rangle + \langle p_B^2(\lambda) \rangle - 1) \quad (105)$$

For Category 2 states with sub-system B involving a local hidden state $\hat{\rho}_\lambda^B$, the key inequality (103) gives

$$|\langle \hat{a}^\dagger \hat{b} \rangle|^2 \leq \frac{1}{4} \sum_\lambda P(\lambda|c) (\langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle) (\langle x_B^2(\lambda) \rangle + \langle p_B^2(\lambda) \rangle - 1) \quad (106)$$

For Category 3 states with neither sub-system involving a local hidden state, the key inequality (103) gives

$$|\langle \hat{a}^\dagger \hat{b} \rangle|^2 \leq \frac{1}{4} \sum_\lambda P(\lambda|c) (\langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle) (\langle x_B^2(\lambda) \rangle + \langle p_B^2(\lambda) \rangle) \quad (107)$$

5.6.3 Strong Correlation Tests for Entanglement and EPR Steering

Using (25), (26), (51) and (50) we can then convert these inequalities to expressions involving number operators, $\hat{N}_C = \hat{c}^\dagger \hat{c}$ ($C = A, B$)

For Category 1 states

$$\begin{aligned} |\langle \hat{a}^\dagger \hat{b} \rangle|^2 &\leq \left\langle \left(\hat{N}_A + \hat{V}_A - \frac{1}{2} \hat{1}_A \right) \otimes \left(\hat{N}_B + \hat{V}_B - \frac{1}{2} \hat{1}_B \right) \right\rangle \\ &= \langle \hat{N}_A \otimes \hat{N}_B \rangle \end{aligned} \quad (108)$$

Hence if

$$|\langle \hat{a}^\dagger \hat{b} \rangle|^2 > \langle \hat{N}_A \otimes \hat{N}_B \rangle \quad (109)$$

the state must be *entangled*. This is the same as the *strong correlation* test originally obtained by Hillery et al [19].

For Category 2 states (with B involving the local hidden state)

$$\begin{aligned} |\langle \hat{a}^\dagger \hat{b} \rangle|^2 &\leq \left\langle \left(\hat{N}_A + \hat{V}_A \right) \otimes \left(\hat{N}_B + \hat{V}_B - \frac{1}{2} \hat{1}_B \right) \right\rangle \\ &= \left\langle \left(\hat{N}_A + \frac{1}{2} \hat{1}_A \right) \otimes \hat{N}_B \right\rangle \end{aligned} \quad (110)$$

Hence if

$$|\langle \hat{a}^\dagger \hat{b} \rangle|^2 > \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle \quad (111)$$

the state must be *EPR steerable*. This is a *strong correlation* test for EPR steering. Note that the condition is harder to satisfy than the test (109) for entanglement, but obviously if (111) is satisfied the state is entangled as well as being EPR steerable. If A involved the LHS then the right side would have been $\langle \hat{N}_A \otimes (\hat{N}_B + \frac{1}{2} \hat{1}_B) \rangle$.

For Category 3 states

$$\begin{aligned} |\langle \hat{a}^\dagger \hat{b} \rangle|^2 &\leq \left\langle \left(\hat{N}_A + \hat{V}_A \right) \otimes \left(\hat{N}_B + \hat{V}_B \right) \right\rangle \\ &= \left\langle \left(\hat{N}_A + \frac{1}{2} \hat{1}_A \right) \otimes \left(\hat{N}_B + \frac{1}{2} \hat{1}_B \right) \right\rangle \end{aligned} \quad (112)$$

where we note that $\hat{N}_A + \frac{1}{2} \hat{1}_A = \hat{a}^\dagger \hat{a} + \frac{1}{2} = (\hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a})/2$. This result is the same as the general result (104) found for all Bell local states.

5.6.4 Hillery-He Spin Variance Tests for Entanglement and EPR Steering

The inequalities (108), (110) and (112) derived above can be put into a more useful form involving *spin operators* - whose mean values and variances can be measured. We use the definitions of the spin operators in SubSection 5.2 (see also Ref. [3])

$$\begin{aligned} |\langle \hat{a}^\dagger \hat{b} \rangle|^2 &= \langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2 \\ \hat{N}_A &= \frac{1}{2} \hat{N} - \hat{S}_z & \hat{N}_B &= \frac{1}{2} \hat{N} + \hat{S}_z \\ \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 &= \frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right) \end{aligned} \quad (113)$$

to find after some straightforward calculations and introducing the variances $\langle \Delta \hat{S}_x^2 \rangle = \langle \hat{S}_x^2 \rangle - \langle \hat{S}_x \rangle^2$ etc the following results. The details are given in Appendix 14.

For Category 1, 2 and 3 states

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle \geq 0 \quad \text{Cat 1 States} \quad (114)$$

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \geq 0 \quad \text{Cat 2 States} \quad (115)$$

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle + \frac{1}{4} \geq 0 \quad \text{Cat 3 States} \quad (116)$$

For Category 2 states with A involving the LHS then the left side would have involved $-\frac{1}{2} \langle \hat{S}_z \rangle$.

The result (114) leads to the same test for *quantum entanglement* as the Hillery spin variance [19] test

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle < 0 \quad (117)$$

This condition can also be written as

$$E_{HZ} = \frac{\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle}{\frac{1}{2} \langle \hat{N} \rangle} < 1 \quad (118)$$

which is the form given in Ref. [18].

The result (115) provides for a test for *EPR steering*

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle < 0 \quad (119)$$

in the case where the LHS occurs in sub-system B . For $\langle \hat{S}_z \rangle = 0$ this test was previously obtained by He et al [18]. If sub-system A involves the LHS then $+\frac{1}{2} \langle \hat{S}_z \rangle$ is replaced by $-\frac{1}{2} \langle \hat{S}_z \rangle$. Since $+\frac{1}{2} \langle \hat{N} \rangle \geq \langle \hat{S}_z \rangle \geq -\frac{1}{2} \langle \hat{N} \rangle$ then $\frac{1}{2} \langle \hat{N} \rangle \geq \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \geq 0$, so as $\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle = \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle + (\frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle)$ and we have just shown that $(\frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle)$ is never negative, then if (119) is satisfied then (117) will also apply. This confirms that EPR steerable states must also be entangled. Since $0 \leq \frac{1}{4} \langle \hat{N} \rangle - \frac{1}{2} \langle \hat{S}_z \rangle \leq \frac{1}{2} \langle \hat{N} \rangle$ it is of course harder to find states where $\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle < \frac{1}{4} \langle \hat{N} \rangle - \frac{1}{2} \langle \hat{S}_z \rangle$ to show EPR steering than merely being $< \frac{1}{2} \langle \hat{N} \rangle$, as would show entanglement. The EPR steering test (119) is a more difficult test to satisfy than the Hillery test for entanglement. In these forms the EPR steering test now allows for *asymmetry* ($\langle \hat{S}_z \rangle \neq 0$). The EPR steering test can also be written as

$$E_{HZ} = \frac{\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle}{\frac{1}{2} \langle \hat{N} \rangle} < \frac{\langle \hat{N}_A \rangle}{\langle \hat{N} \rangle} \quad (120)$$

after substituting $\langle \hat{N} \rangle = \langle \hat{N}_A \rangle + \langle \hat{N}_B \rangle$ and $\langle \hat{S}_z \rangle = (\langle \hat{N}_B \rangle - \langle \hat{N}_A \rangle)/2$, which is consistent with the result $E_{HZ} < 1/2$ given in Ref. [18] for $\langle \hat{S}_z \rangle = 0$. As $\langle \hat{N}_A \rangle \leq \langle \hat{N} \rangle$ clearly E_{HZ} may be required to be much smaller than unity to establish EPR steering as distinct from just confirming entanglement.

The inequality (116) applies for *all* quantum states, so no useful test for *Bell non-locality* occurs.

An alternative derivation of these results is given in Appendix 14.

5.7 Tests for EPR Steering - Quadrature Squeezing Test ?

Here we consider whether a further test that has been shown to be sufficient to demonstrate quantum entanglement (violation of Category 1) (see Ref. [3] for details) is also valid for demonstrating EPR steering. This test is squeezing in any of the two mode quadrature observables.

5.7.1 Two Mode Quadrature Squeezing Test

For Category 1 states it has been shown (see [3]) that if there is squeezing in any of the two mode quadrature operators

$$\langle \Delta X_\theta(\pm)^2 \rangle < \frac{1}{2} \quad \text{or} \quad \langle \Delta P_\theta(\pm)^2 \rangle < \frac{1}{2} \quad (121)$$

then the state is entangled - that is, it is not in Category 1. Due to the Heisenberg uncertainty principle $\langle \Delta X_\theta(\pm)^2 \rangle \langle \Delta X_\theta(\pm)^2 \rangle \geq 1/4$ only one of the pair of quadrature operators is squeezed.

However we have shown for Category 2 states (see Eq. (89)) that

$$\langle \Delta X_\theta(\pm)^2 \rangle = \langle \Delta X_\theta(\pm)^2 \rangle = \frac{1}{2} \langle N \rangle + \frac{1}{2} \quad (122)$$

and as the right side is never less than one half, it follows that two mode quadrature squeezing in either $X_\theta(\pm)$ or $P_\theta(\pm)$ shows that the state cannot be in Category 2. Thus Eq. (121) provides a test for EPR steering.

6 Spin Observables Test for Bell Non-Locality

All Bell local states satisfy the well-known Bell inequality [4]. This is based on four sets of joint measurements involving a pair of observables A_1, A_2 for sub-system A and B_1, B_2 for sub-system B . The standard case treated is based on the requirement that the measurement outcomes for any of the four observables is either $+1$ or -1 - as for example when *polarisations* of photon modes are observed with outcomes \uparrow and \rightarrow (or \curvearrowright and \curvearrowleft). In this situation the form given by Clauser et al [14] for *Bell's inequality* is

$$|S| \leq 2 \quad (123)$$

where

$$S = \langle A_1 \times B_1 \rangle_{LHV} + \langle A_1 \times B_2 \rangle_{LHV} + \langle A_2 \times B_1 \rangle_{LHV} - \langle A_2 \times B_2 \rangle_{LHV} \quad (124)$$

The minus sign can actually be attached to any one of the four terms. A proof of this important result that all Bell local states must satisfy is given in [21] and in Ref [2].

However, for systems of identical massive bosons the observables of interest may not be restricted to those with only two possible outcomes. For example, in the case of two sub-systems consisting of modes a and b , the observables may be quadrature amplitudes x_A, p_A for sub-system A and x_B, p_B for sub-system B . Here the measurement outcomes would be *continuous* numbers from $-\infty$ to $+\infty$. Another example is for the case of a system consisting of four modes a_1, a_2 which constitute sub-system A and b_1, b_2 which constitute sub-system B . As in a situation treated in Ref [18], spin operators can be defined for each sub-system $\hat{S}_{Ax} = (\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2)/2$, $\hat{S}_{Ay} = (\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2)/2i$, $\hat{S}_{Az} = (\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1)/2$ for sub-system A and $\hat{S}_{Bx} = (\hat{b}_2^\dagger \hat{b}_1 + \hat{b}_1^\dagger \hat{b}_2)/2$, $\hat{S}_{By} = (\hat{b}_2^\dagger \hat{b}_1 - \hat{b}_1^\dagger \hat{b}_2)/2i$, $\hat{S}_{Bz} = (\hat{b}_2^\dagger \hat{b}_2 - \hat{b}_1^\dagger \hat{b}_1)/2$ for sub-system B . For simulations eigenstates $|N_A\rangle \otimes |N_B\rangle$ of the sub-system number operators $\hat{N}_A = \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_1$ and $\hat{N}_B = \hat{b}_2^\dagger \hat{b}_2 + \hat{b}_1^\dagger \hat{b}_1$, the measured outcomes of any of the A spin components are *quantized* and range from $-N_A/2$ to $+N_A/2$ in integer steps, and range from $-N_B/2$ to $+N_B/2$ in integer steps for the B spin components.

We therefore need to *extend* the Bell inequality to include cases where the measured outcomes are either continuous or quantized. Here we just treat the *quantized* case for observables that are *spin components*. Our approach is based on the treatment set out in Appendix G of Ref. [2].

6.1 Extension of Bell Inequality - Spin Components

From the basic LHV result (19) for the mean value $\langle A_i \times B_j \rangle_{LHV}$ we have (here $\lambda \rightarrow \xi$, $P(\lambda|c) \rightarrow P(\xi)$, $\Omega_{Ai} \rightarrow A_i$, $\Omega_{Bj} \rightarrow B_j$, $\langle \Omega_{Ai}(\lambda, c) \rangle \rightarrow \langle A_i(\xi) \rangle$,

$\langle \Omega_{Bj}(\lambda, c) \rightarrow \langle B_j(\xi) \rangle$ for short)

$$\begin{aligned}
& \frac{1}{4} \langle N_A \rangle \langle N_B \rangle (\langle A_2 \times B_1 \rangle_{LHV} - \langle A_2 \times B_2 \rangle_{LHV}) \\
&= \int d\xi P(\xi) (\langle A_2(\xi) \rangle \langle B_1(\xi) \rangle - \langle A_2(\xi) \rangle \langle B_2(\xi) \rangle) \frac{1}{4} \langle N_A \rangle \langle N_B \rangle \\
&= \int d\xi P(\xi) (\langle A_2(\xi) \rangle \langle B_1(\xi) \rangle) \left(\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_2(\xi) \rangle \right) \\
&\quad - \int d\xi P(\xi) (\langle A_2(\xi) \rangle \langle B_2(\xi) \rangle) \left(\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_1(\xi) \rangle \right)
\end{aligned} \tag{125}$$

In the case of *spin components* all the quantities $\langle A_i(\xi) \rangle$ are bounded by $+\frac{1}{2} \langle N_A \rangle$ or $-\frac{1}{2} \langle N_A \rangle$, and all the quantities $\langle B_j(\xi) \rangle$ are bounded by $+\frac{1}{2} \langle N_B \rangle$ or $-\frac{1}{2} \langle N_B \rangle$. To show this in the case of the A sub-system spin components, we note that as every spin component commutes with the sub-system number operator, then LHVT provides for a joint probability $P(s_{Ai}^k, n_A | S_{Ai}, N_A, \xi, c)$ that measurement of spin component S_{Ai} and particle number N_A leads to outcomes s_{Ai}^k and n_A when the hidden variables ξ occur in preparation c . As LHVT is to be consistent with quantum theory the outcomes s_{Ai}^k are quantized and range from $-n_A/2$ to $+n_A/2$ in integer steps, whilst the n_A are also quantized and given by $0, 1, 2, \dots$. Hence for the mean value of the spin component S_{Ai} when the hidden variables are ξ

$$\begin{aligned}
\langle S_{Ai}(\xi) \rangle &= \sum_{k, n_A} s_{Ai}^k P(s_{Ai}^k, n_A | S_{Ai}, N_A, \xi, c) \\
|\langle S_{Ai}(\xi) \rangle| &\leq \sum_{k, n_A} |s_{Ai}^k| P(s_{Ai}^k, n_A | S_{Ai}, N_A, \xi, c) \\
&\leq \sum_{k, n_A} \frac{n_A}{2} P(s_{Ai}^k, n_A | S_{Ai}, N_A, \xi, c) \\
&\leq \sum_{n_A} \frac{n_A}{2} P(n_A | N_A, \xi, c) \\
&\leq \frac{1}{2} \langle N_A(\xi) \rangle
\end{aligned} \tag{126}$$

since for a given n_A the outcomes s_{Ai}^k for the measurement of S_{Ai} all have magnitudes $\leq n_A/2$, and the sum $\sum_k P(s_{Ai}^k, n_A | S_{Ai}, N_A, \xi, c)$ for a given n_A gives the single measurement probability $P(n_A | N_A, \xi, c)$. Hence

$$\begin{aligned}
\int d\xi P(\xi) |\langle S_{Ai}(\xi) \rangle| &\leq \int d\xi P(\xi) \frac{1}{2} \langle N_A(\xi) \rangle \\
&\leq \frac{1}{2} \langle N_A \rangle
\end{aligned} \tag{127}$$

This shows that each $|\langle S_{Ai}(\xi) \rangle|$ must be $\leq \frac{1}{2} \langle N_A \rangle$, because if $|\langle S_{Ai}(\xi) \rangle|$ was greater than $\frac{1}{2} \langle N_A \rangle$, then $\int d\xi P(\xi) |\langle S_{Ai}(\xi) \rangle| > \frac{1}{2} \langle N_A \rangle \int d\xi P(\xi) = \frac{1}{2} \langle N_A \rangle$,

since all the $P(\xi) \geq 0$ and $\int d\xi P(\xi) = 1$. Thus after applying a similar approach for sub-system B we have

$$|\langle S_{Ai}(\xi) \rangle| \leq \frac{1}{2} \langle N_A \rangle \quad |\langle S_{Bj}(\xi) \rangle| \leq \frac{1}{2} \langle N_B \rangle \quad (128)$$

Hence the expressions $(\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_2(\xi) \rangle)$ and $(\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_1(\xi) \rangle)$ are never negative. Taking the modulus of the left side leads to an equality

$$\begin{aligned} & \frac{1}{4} \langle N_A \rangle \langle N_B \rangle (|\langle A_2 \times B_1 \rangle_{LHV} - \langle A_2 \times B_2 \rangle_{LHV}|) \\ & \leq \int d\xi P(\xi) (|\langle A_2(\xi) \rangle| |\langle B_1(\xi) \rangle| (\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_2(\xi) \rangle) \\ & \quad + \int d\xi P(\xi) (|\langle A_2(\xi) \rangle| |\langle B_2(\xi) \rangle| (\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_1(\xi) \rangle)) \\ & \leq \frac{1}{4} \langle N_A \rangle \langle N_B \rangle \int d\xi P(\xi) (\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_2(\xi) \rangle) \\ & \quad + \frac{1}{4} \langle N_A \rangle \langle N_B \rangle \int d\xi P(\xi) (\frac{1}{4} \langle N_A \rangle \langle N_B \rangle \pm \langle A_1(\xi) \rangle \langle B_1(\xi) \rangle) \end{aligned} \quad (129)$$

where we have again used the results that $|\langle A_2(\xi) \rangle| \leq \frac{1}{2} \langle N_A \rangle$, $|\langle B_1(\xi) \rangle| \leq \frac{1}{2} \langle N_B \rangle$ and $|\langle B_2(\xi) \rangle| \leq \frac{1}{2} \langle N_B \rangle$.

Dividing out the factor $\frac{1}{4} \langle N_A \rangle \langle N_B \rangle$ on each side and using $\int d\xi P(\xi) = 1$ gives the inequality

$$\begin{aligned} & |\langle A_2 \times B_1 \rangle_{LHV} - \langle A_2 \times B_2 \rangle_{LHV}| \\ & \leq \frac{1}{2} \langle N_A \rangle \langle N_B \rangle \pm (\int d\xi P(\xi) \langle A_1(\xi) \rangle \langle B_2(\xi) \rangle + \int d\xi P(\xi) \langle A_1(\xi) \rangle \langle B_1(\xi) \rangle) \\ & = \frac{1}{2} \langle N_A \rangle \langle N_B \rangle \pm (\langle A_1 \times B_2 \rangle_{LHV} + \langle A_1 \times B_1 \rangle_{LHV}) \end{aligned} \quad (130)$$

Hence since $|\langle A_1 \times B_2 \rangle_{LHV} + \langle A_1 \times B_1 \rangle_{LHV}| = +(\langle A_1 \times B_2 \rangle_{LHV} + \langle A_1 \times B_1 \rangle_{LHV})$ or $-(\langle A_1 \times B_2 \rangle_{LHV} + \langle A_1 \times B_1 \rangle_{LHV})$ we have

$$|\langle A_2 \times B_1 \rangle_{LHV} - \langle A_2 \times B_2 \rangle_{LHV}| \pm |\langle A_1 \times B_2 \rangle_{LHV} + \langle A_1 \times B_1 \rangle_{LHV}| \leq \frac{1}{2} \langle N_A \rangle \langle N_B \rangle \quad (131)$$

But since $|X - Y| \leq |X| + |Y|$ we see that from the + version of the last inequality that

$$|\langle A_2 \times B_1 \rangle_{LHV} - \langle A_2 \times B_2 \rangle_{LHV} + \langle A_1 \times B_2 \rangle_{HVT} + \langle A_1 \times B_1 \rangle_{HVT}| \leq \frac{1}{2} \langle N_A \rangle \langle N_B \rangle \quad (132)$$

This is a *Bell inequality* for the case where the A_i and B_j are components of spin observables for the respective sub-systems and the N_A and N_B are the sub-system number observables.

Hence if

$$|\langle A_2 \times B_1 \rangle_{LHV} - \langle A_2 \times B_2 \rangle_{LHV} + \langle A_1 \times B_2 \rangle_{HVT} + \langle A_1 \times B_1 \rangle_{HVT}| > \frac{1}{2} \langle N_A \rangle \langle N_B \rangle \quad (133)$$

we have a sufficient *test* that the state is *Bell non-local*.

Note that for the case where there is exactly one particle in each sub-system so that $\langle N_A \rangle = \langle N_B \rangle = 1$, the right side of (133) equals $1/2$. In this case the possible measured outcomes for any of the spin components is $\pm 1/2$ - which means that as in the standard Bell inequality situation there are just two possible outcomes for measuring the sub-system observables. However, as now the outcomes are $\pm 1/2$ rather than ± 1 the right side of the Bell inequality becomes $1/2$ rather than 2 as in the standard situation.

Interchanging $A_2 \leftrightarrow A_1$ and repeating the derivation gives $|\langle A_1 \times B_1 \rangle_{HVT} - \langle A_1 \times B_2 \rangle_{LHV} + \langle A_2 \times B_2 \rangle_{LHV} - \langle A_2 \times B_1 \rangle_{LHV}| \leq \frac{1}{2} \langle N_A \rangle \langle N_B \rangle$, which is another Bell inequality. Interchanging $B_1 \leftrightarrow B_2$ and repeating the derivation gives $|\langle A_2 \times B_2 \rangle_{LHV} - \langle A_2 \times B_1 \rangle_{LHV} + \langle A_1 \times B_1 \rangle_{LHV} + \langle A_1 \times B_2 \rangle_{LHV}| \leq \frac{1}{2} \langle N_A \rangle \langle N_B \rangle$, and interchanging $A_2 \leftrightarrow A_1$ and $B_1 \leftrightarrow B_2$ and repeating the derivation gives $|\langle A_1 \times B_2 \rangle_{LHV} - \langle A_1 \times B_1 \rangle_{LHV} + \langle A_2 \times B_1 \rangle_{LHV} + \langle A_2 \times B_2 \rangle_{LHV}| \leq \frac{1}{2} \langle N_A \rangle \langle N_B \rangle$. Thus the minus sign can be attached to any one of the four terms.

6.2 State Violating Bell Inequality for Spin Components

For the standard Bell inequality in which measurements of the sub-system observables lead to only two possible outcomes, it is well-known that the Bell singlet state for two spin $1/2$ sub-systems is associated with violations of the Bell inequality and therefore this Bell singlet state is Bell non-local. It would therefore be of interest to find a Bell non-local state for the present case where spin components for the two sub-systems are measured and where the numbers of particles in each sub-system may be macroscopic.

We need to evaluate the mean values of the form

$$\begin{aligned} & \left\langle \underline{u}_A \cdot \hat{\underline{S}}_A \otimes \underline{v}_B \cdot \hat{\underline{S}}_B \right\rangle \\ & \left\langle \hat{N}_A \otimes \hat{1}_B \right\rangle, \left\langle \hat{1}_A \otimes \hat{N}_B \right\rangle \end{aligned} \quad (134)$$

where \underline{u}_A and \underline{v}_B are *unit vectors* for the proposed Bell non-local state (YET TO BE FOUND).and see if there are choices for these unit vectors so that (133) applies.

7 Summary and Conclusion

We have reviewed two possible classification schemes for the quantum states of bipartite composite systems. In the first (Quantum Theory Classification Scheme) the states are classified as being either quantum separable or quantum entangled. In the second (Local Hidden Variable Theory Classification Scheme) the states are initially classified as being Bell local or Bell non-local. The Bell non-local states are quantum entangled and EPR steerable - these are listed as Category 4 states. However, the Bell local states can be divided up into three categories depending on whether both, one or neither of the sub-system single measurement probability is given by a quantum theory expression involving a sub-system density operator. The Category 1 states (both) are the same as the quantum separable states and are non-entangled, LHS states and non-steered. The Category 2 states (one) are quantum entangled LHS states (LHS) and are non-steerable. The Category 3 (neither) states are quantum entangled and EPR steerable.

A detailed study of how observables are treated in terms of quantum theory and local hidden variable theories has been carried out, including how the two approaches are related and how to replace quantum operators for observables with classical entities. For systems involving identical bosons the mode annihilation, creation operators are replaced by quadrature amplitudes.

Tests for EPR steering (EPR entanglement) based on violation of the LHS model have been examined. Such tests were obtained based on whether the Bloch vector is in the xy plane (Bloch vector test), on the mean value of $\langle a^\dagger b \rangle$ (correlation test) and on whether there is spin squeezing in any of the spin components S_x , S_y or S_z (spin squeezing S_z test). The Hillery spin variance test based on the sum of variances in S_x and S_y also demonstrates EPR steering. In addition, two mode quadrature squeezing also provides a test for EPR steering. A new strong correlation test for EPR steering was found, as well as a test involving the sum of variances in S_x and S_y , but now containing a different multiple of the mean value for N along with a term involving the mean value for S_z . This allows for asymmetry and is a stronger version of the Hillery spin variance test. No EPR steering test based on the difference between the variances of the number difference and number sum was found. We note that most of the tests were based on applying the super-selection rules for the total particle number as well as that for the local particle number for the sub-system LHS. Exceptions are the Hillery spin variance test and its generalisation involving the mean value for S_z as well as the strong correlation test. Finally, a new test for Bell non-locality for macroscopic systems of identical bosons was obtained for the case where the sub-system observables are spin components, whose measured outcomes are no longer restricted to ± 1 .

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9 Appendix A - Hidden Variable Theory Requirements

As noted in the Introduction, hidden variable theories were based on Einstein's hypothesis that although quantum theory is *correct*, it was *incomplete* in that it could be underpinned by an underlying *classical* deterministic theory that described the true *reality* and which was built on *hidden variables*. Such an approach has been described above in the context of providing an alternative theory of the *joint probabilities* for the *outcomes* of measurements of pairs of *observables* on *separate sub-systems* of a *composite* quantum system. This aspect of hidden variable theory focused on the *key quantum features* of measurement outcomes being governed by *probabilities* in general, and the unusual features of these probabilities associated with *entanglement* and *Bell non-locality* for composite quantum systems.

Basing the two Categories 2 and 3 on LHV theories in which one or two of the sub-system measurement probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ for particular preparation processes are *not* determinable via any well-defined theory that actually can be used to *derive* the probability expressions from the evolution of the hidden variables associated with a *known* initial state to those associated with the *prepared* state may seem a rather questionable approach from the general scientific methodology point of view. However, the point of view involved is based on seeing if *any* LHV theory can account for the measurement outcomes *irrespective* of whether a *consistent* and *complete* LHV dynamical theory is actually *available*. The aim is to see if all LHV can be ruled out, which then would make it *unnecessary* to try to develop a fully predictive version. In contrast with all local hidden variable theories, quantum theory *does* provide an actual consistent and complete dynamical theory both for determining the density operator $\hat{\rho}$ associated with a given preparation process, and for calculating the measurement probabilities via (6) and (7). As previously noted, Einstein et al [8] did not doubt the *correctness* of such quantum predictions, but rather asked the question whether the joint measurement probabilities could be accounted for by some sort of *underlying* realistic theory - such as the local hidden variable theories considered here.

Although the possible Bell inequality violations are important situations for consideration regarding whether hidden variable theories are compatible with quantum theory, we wish to point out that there is actually *more* than just one key feature of quantum theory that a hidden variable theory would need to account for. The hidden variable theory approach is even *harder* to implement than just dealing with Bell locality. As we will see, several of these features are *difficult* to account for via a hidden variable theory. We begin by listing a number of *key features* of quantum theory.

1. *Probabilities for Measurement Outcomes.* Outcomes only predictable probabilistically, in contrast to deterministic outcomes in classical theory. Born's rules based on square of probability amplitudes. Classically forbidden events (quantum tunneling) may occur that are forbidden in classical physics.

2. *Quantum Superposition.* States can occur that are superpositions of other states leading to superposition of probability amplitudes. Quantum constructive and destructive interference effects in outcome probability amplitudes.

3. *Quantum Entanglement.* Feature may occur in certain states for composite quantum systems. Effects such as violations of Bell inequalities. Schrodinger Cat paradox.

4. *Quantisation.* The possible measured outcomes for certain physical observables only take on certain discrete values, with values in between never occurring. Often the quantised outcomes depend on Planck's constant, which does not appear in classical theories. Quantum numbers often related to irreducible representations of a symmetry group.

5. Lack of *Simultaneous Measurement* Outcomes. Certain pairs of physical observables are such that carrying out a precise measurement for one leads to imprecise measurement outcomes for the other, as exhibited via Heisenberg Uncertainty Principle inequalities for the variances of the measurements - again often involving Planck's constant.

6. *Wave-Particle Duality.* A quantum system sometimes behaves like a classical particle, sometimes like a classical wave - but never both in the same experiment. Related to Heisenberg Uncertainty Principle.

7. *Non-Classical* Physical Quantities. New physical quantities occur which are absent from classical theories (parity, spin angular momentum,...). Often these quantities are related to symmetries of the system.

8. *Symmetrisation Principles* for *Identical Particles.* The quantum state can at most change sign when identical particles are interchanged. Related to permutation symmetry and seen in phenomena such as the periodic table and Bose-Einstein condensation.

9. *Creation and Destruction* of Matter. The basic particles do not always have a permanent existence when the relativistic regime of quantum theory is involved, unlike in classical theory. Associated with quantum theory of fields.

10. Dynamical *Evolution* of *Quantum States.* States evolve in accordance with non-classical equations, such as Liouville-von Neumann equation. Involves Planck's constant and is first order in time in contrast to classical equations such as Newton's law, which is second order in time.

11. *Conservation Laws.* Measurement outcome probabilities for certain physical quantities may not change during state evolution. Related to symmetry of system Hamiltonian.

12. *Classical Limit.* For some types of quantum system (macroscopic systems,...) and/or for certain quantum states (angular momentum large compared to Planck's constant,...) the system behaviour approaches that predicted by classical physics. Feynman path integral quantum treatment leads to negligible probabilities for trajectories off the classical path, Ehrenfest's theorem, ...

There is a *strong* link between the key features of quantum theory and the *mathematical structure* that is applied in quantum theory. Representing pure *quantum states* by vectors in a *Hilbert space* leads naturally to the *quantum superposition* feature. *Tensor products* of Hilbert spaces being used for composite quantum systems is closely linked to *quantum entanglement* and the *symmetri-*

sation principle. Representing *physical quantities* by *linear Hermitian operators* is linked to the possible measured outcomes being associated with one of the main characteristics of such operators - the spectrum of *real eigenvalues*. In addition such sets of eigenvalues often exhibit the *quantisation* effects associated with the outcomes of measurements in quantum theory. Representing physical quantities by operators also facilitates the presence of *non-classical* physical quantities as new linear Hermitian operators for the system are identified. *Symmetry properties* are linked to the presence of groups of *unitary operators* and *conserved physical quantities* are associated with the *generators* of these operator *groups*. The lack of *simultaneous measurability* of certain pairs of physical quantities is related to the absence of *simultaneous eigenvectors* for *non-commuting operators*. Finally, the *measurement probabilities* can be expressed in terms of mathematical processes in the Hilbert space that involve the vectors representing the quantum states and the *projector operators* associated with the eigenvalues and *eigenvectors* of the operator representing the observable quantity.

In contrast, the link between the mathematical structures generally used in trying to formulate an underlying hidden variable theory that could account for all these quantum features is rather *weak*. This is often essentially the same as that used in classical physics involving *time dependent functions* or *fields* to represent the *classical state*, with *physical quantities* introduced as *mathematical operations* on these *functions* - such as the velocity being the time derivative of the function specifying position. Although no one can categorically rule out hidden variable theories as being incapable of accounting even for most of the above quantum features (the violation of Bell inequalities is not regarded as being compatible with hidden variable theory), it does at least seem to be the case that the task would be a very difficult one - probably requiring the introduction of a *novel* mathematical structure to describe such a theory. To *highlight* some of the *issues*: - How would quantisation appear from the theory? How would symmetrisation effects based on indistinguishable particles be accounted for when hidden variable theories are supposed to enable such particles to be distinguished by their differing trajectories? Where does Planck's constant come from?

Given the flexibility in the range of possible hidden variable theories it is possible to generate *some* of these *quantum features* by just requiring the HVT to be subject to certain *constraints*. For example, quantisation of an angular momentum component J_α could be achieved by requiring $P(j_\alpha|J_\alpha, c, \lambda)$ to be zero unless the outcome j_α is an integer or half integer times \hbar . This would be rather *ad hoc*. However, of all the quantum features the one that has rightly been identified as the key feature which rules out HVT is the *violation* of a *Bell inequality*. These inequalities are constructed in terms of the most general unconstrained form of local HVT, so their violation must rule out a HVT interpretation.

10 Appendix B - Idea of EPR Steering

In this Appendix we consider for reasons of completeness the physical idea behind EPR steering, as presented in the papers [5], [6] and [7].

We can derive expressions within LHV theory for the *conditional probabilities* defined in (5). These expressions apply for all three Bell local categories considered here. We will focus on LHS states, which in terms of our LHVCS may be either in Category 1 or Category 2. We will initially consider the latter.

In the case of *Category 2* states (which are *LHS states*) we obtain from (41) and (5)

$$P(\beta|\Omega_B||\alpha, \Omega_A, c) = \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \text{Tr}_B((\hat{\Pi}_{\beta}^B) \hat{\rho}^B(\lambda))}{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c)} \quad (135)$$

using (17) and (43).

It is also important to realise that these LHS model states are still related to an overall quantum state, but one which is *non-separable* since we cannot derive the density operator (33) for separable states from Category 2 expression (41) for the joint probability. For Category 2 LHS states, $P(\alpha|\Omega_A, c, \lambda)$ is *not* given by a quantum expression. However, as in [6], [7] we can relate the quantities in the LHS model (41) to a density operator for sub-system B that is *conditional* on the results for measurements on sub-system A .

From (7) the *quantum theory* result for the probability that measurement of observable Ω_A results in outcome α is given by

$$P(\alpha|\Omega_A, \rho) = \text{Tr}((\hat{\Pi}_{\alpha}^A \otimes \hat{1}^B) \hat{\rho}) \quad (136)$$

where $\hat{\rho}$ is the density operator for the overall quantum state (the preparation symbol c is left out for simplicity). In the Copenhagen interpretation of quantum theory the *normalised* state that is produced as a *result* of this measurement is the *conditional state*

$$\hat{\rho}_{cond}(\alpha|\Omega_A, \rho) = (\hat{\Pi}_{\alpha}^A \otimes \hat{1}^B) \hat{\rho} (\hat{\Pi}_{\alpha}^A \otimes \hat{1}^B) / P(\alpha|\Omega_A, \rho) \quad (137)$$

This state has a trace of unity, as required. To confirm that $\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)$ *does* lead to the correct quantum expression for the *conditional probability* $P(\beta|\Omega_B||\alpha|\Omega_A, \rho)$ that measurement of Ω_B in sub-system B will result outcome β *given* that measurement of Ω_A resulted in outcome α based on the quantum state $\hat{\rho}$, we calculate the probability of that measurement of Ω_B in sub-system B will result outcome β for the quantum state $\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)$.

$$\begin{aligned} P(\beta|\Omega_B, \rho_{cond}) &= \text{Tr}((\hat{1}^A \otimes \hat{\Pi}_{\beta}^B) \hat{\rho}_{cond}(\alpha|\Omega_A, \rho)) \\ &= \text{Tr}((\hat{\Pi}_{\alpha}^A \otimes \hat{\Pi}_{\beta}^B) \hat{\rho} (\hat{\Pi}_{\alpha}^A \otimes \hat{1}^B)) / P(\alpha|\Omega_A, \rho) \\ &= \text{Tr}((\hat{\Pi}_{\alpha}^A \otimes \hat{\Pi}_{\beta}^B) \hat{\rho}) / P(\alpha|\Omega_A, \rho) \\ &= P(\alpha, \beta|\Omega_A, \Omega_B, \rho) / P(\alpha|\Omega_A, \rho) \\ &= P(\beta|\Omega_B||\alpha|\Omega_A, \rho) \end{aligned} \quad (138)$$

using the cyclic properties of the trace and $(\hat{\Pi}_\alpha^A)^2 = \hat{\Pi}_\alpha^A$, with the last line (see (5)) following from *Bayes' theorem*. This confirms the status of $\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)$.

The *physical* concept of *steering* has been discussed in several papers, including [5], [6] and [7] and was originally introduced by Schrodinger [9] following the important EPR paper [8]. The key idea is that when a measurement of Ω_A is made on sub-system A resulting in outcome α (the bipartite quantum state prepared being ρ) this results in both the overall quantum state changing to a new conditioned state $\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)$ (given in Eq. (137)) and hence the *post-measurement* state describing sub-system B changing to

$$\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)^B = Tr_A(\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)) \quad (139)$$

from its *pre-measurement* state $\hat{\rho}^B = Tr_A(\hat{\rho})$ given by the *reduced density operator* (8). This strange quantum effect allows for an experiment carried out on sub-system A to instantly change (or "steer") the quantum state for sub-system B into a new quantum state, even when the two sub-systems are localised in well-separated spatial regions and the experimenter on A may have no direct access to sub-system B . For those who accept the Copenhagen interpretation of quantum theory there is nothing really strange involved. Quantum states merely specify all that can be known about the physical state (and no distinction between "physical state" and "quantum state" is made), so as the measurement of Ω_A has led to a particular outcome α our knowledge about the state has changed, and hence the quantum state for both the overall system and its sub-systems should change accordingly. Using quantum theory we can obtain an explicit formula for $\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)^B$ and this is

$$\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)^B = \sum_{\beta l, \gamma n} |B\beta l\rangle \langle B\gamma n| \left(\sum_i \rho_{A\alpha i, B\beta l :: A\alpha i, B\gamma n} \right) \quad (140)$$

where the original density operator ρ is expressed in terms of orthonormal basis states $|A\alpha i\rangle \otimes |B\beta n\rangle$ that are eigenstates for $\hat{\Omega}_A$ and $\hat{\Omega}_B$, with $i = 1, 2, \dots, d_\alpha$ and $n = 1, 2, \dots, d_\beta$ allowing for degeneracy.

We can also show that the sum of the conditional density operators $\hat{\rho}_{cond}(\alpha|\Omega_A, \rho)^B$ each weighted by the probability $P(\alpha|\Omega_A, \rho)$ for the measurement outcome α for Ω_A gives the reduced density operator $\hat{\rho}^B$ associated with the original state ρ . This result is not surprising, since carrying out the measurement of any choice of Ω_A and then discarding the results would be described by reduced density operator.

$$\sum_\alpha P(\alpha|\Omega_A, \rho) \hat{\rho}_{cond}(\alpha|\Omega_A, \rho)^B = \hat{\rho}^B = Tr_A \hat{\rho} \quad (141)$$

The proofs of (140) and (141) are straightforward.

Thus, we have seen how according to quantum theory the quantum state describing sub-system B changes as a result of measuring Ω_A on sub-system A and obtaining outcome α . Furthermore, we have obtained quantum theory expressions (138) for the conditional probability $P(\beta|\Omega_B, \rho_{cond})$ for measurement of Ω_B on sub-system B and obtaining outcome β when measurement of Ω_A on

sub-system A resulted in outcome α and (140) for the quantum state describing sub-system B . The question then is: Although quantum theory gives the correct results for the conditional probability $P(\beta|\Omega_B, \rho_{cond})$, can the same results *also* be explained in a local hidden variable theory?

Following the operational definition for steering in Refs. [5], [6] and [7], the quantum state ρ is only considered to be *EPR steerable* when the conditional probability $P(\beta|\Omega_B||\alpha, \Omega_A, c)$ can *not* be explained via a local hidden variable theory. For the LHS cases of Category 1 and Category 2 states we will see that a LHV theory explanation applies. We consider what expression for a density operator for sub-system B would give the LHS result for the conditional probability $P(\beta|\Omega_B||\alpha, \Omega_A, c)$ for measurement of Ω_B to have outcome β , given that measurement of Ω_A has outcome α and the preparation process is c . In the case of Category 2 states we use Eqs. (41) and (43) in conjunction with (5) and (17) to find

$$P(\beta|\Omega_B||\alpha, \Omega_A, c) = \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) Tr_B((\hat{\Pi}_{\beta}^B) \hat{\rho}^B(\lambda)) P(\lambda|c)}{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c)} \quad (142)$$

We then *define a new* normalised quantum state $\hat{\rho}_{cond}^B(\alpha|\Omega_A, c)$ for sub-system B by the expression

$$\begin{aligned} \hat{\rho}_{cond}^B(\alpha|\Omega_A, c) &= \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \hat{\rho}^B(\lambda) P(\lambda|c)}{Tr_B \left(\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \hat{\rho}^B(\lambda) P(\lambda|c) \right)} \\ &= \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \hat{\rho}^B(\lambda) P(\lambda|c)}{\left(\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c) \right)} \end{aligned} \quad (143)$$

It is to be noted that this state for sub-system B involves local HVT and not quantum expressions for the measurement probabilities $P(\alpha|\Omega_A, c, \lambda)$ for sub-system A . We then see from (7) that for *this state* the probability for measurement of Ω_B to have outcome β is given by

$$\begin{aligned} Tr_B(\hat{\Pi}_{\beta}^B \hat{\rho}_{cond}^B(\alpha|\Omega_A, c)) &= \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) Tr_B((\hat{\Pi}_{\beta}^B) \hat{\rho}^B(\lambda)) P(\lambda|c)}{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c)} \\ &= P(\beta|\Omega_B||\alpha, \Omega_A, c) \end{aligned} \quad (144)$$

which is the same as (135) obtained for the Category 2 states (which are LHS states). Thus the sub-system B quantum state (143) has been constructed purely from the Category 2 LHS model probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\lambda|c)$, together with the LHS model quantum state $\hat{\rho}^B(\lambda)$ - which is a possible quantum

state for sub-system B based on hidden variables λ . The sub-system B quantum state $\hat{\rho}_{cond}^B(\alpha|\Omega_A, c)$ in (143) determines the correct probability for measurement of Ω_B to have outcome β . The same analysis would apply to the LHS states in Category 1, the only difference being that $P(\alpha|\Omega_A, c, \lambda)$ would be replaced by $P_Q(\alpha|\Omega_A, c, \lambda)$ in terms of our notation. So in both of these cases there could be a hidden state $\hat{\rho}^B(\lambda)$ associated with hidden variables that could explain (along with suitable choices for $P(\alpha|\Omega_A, c, \lambda)$ and $P(\lambda|c)$) the measurements on sub-system B . The treatment however does not apply to the quantum states in Category 3, where the LHV model in Eq.(42) does *not* include a quantum state $\hat{\rho}^B(\lambda)$ for sub-system B . Hence, the conditional probability $P(\beta|\Omega_B||\alpha, \Omega_A, c)$ *can* be explained via the *LHS model* for both Category 1 and Category 2 states, showing that the Category 1 and Category 2 quantum states are *non-steerable*. However, the Category 3 states are *EPR steerable*.

11 Appendix C - Werner States

As examples of the three categories of Bell local states we may consider the states introduced by Werner [11] as $U \otimes U$ invariant states $((\hat{U} \otimes \hat{U}) \hat{\rho}_W (\hat{U}^\dagger \otimes \hat{U}^\dagger) = \hat{\rho}_W$, where \hat{U} is any *unitary* operator) for two d dimensional sub-systems. Depending on the parameter η (or ϕ) the Werner states, may be separable or entangled. They may also be Bell local in one of the three categories described above, or they may be Bell non-local. The density operator for the *Werner states* is given by

$$\begin{aligned} \hat{\rho}_W &= (d^3 - d)^{-1} [(d - \phi) \hat{1} + (d\phi - 1) \hat{V}] \\ &= \left(\frac{(d-1+\eta)}{(d-1)} \right) \frac{\hat{1}}{d^2} - \left(\frac{\eta}{(d-1)} \right) \frac{\hat{V}}{d} \end{aligned} \quad (145)$$

where $\hat{1}$ is the *unit* operator and \hat{V} is the *flip* operator ($\hat{V}(|\psi\rangle \otimes |\chi\rangle) = (|\chi\rangle \otimes |\psi\rangle)$). The two expressions are interconvertable with $\phi = (1 - (d+1)\eta)/d$. Werner had showed that if $\eta < 1/(d+1)$ (or $\phi > 0$) the state $\hat{\rho}_W$ is separable, for $\eta > 1/(d+1)$ (or $\phi < 0$) the state was entangled. Thus Werner states with $\eta < 1/(d+1)$ or $\phi > 0$ are separable and correspond to the states shown in Region A in all Figures 1, 2 and 3. Wiseman et al [5] considered the above categories for such Werner states and determined the parameter boundaries for the various categories. These results are shown in Figure 2 (taken from Figure 1 in Ref [5]), where the parameter regimes for the various categories of quantum states are explained.

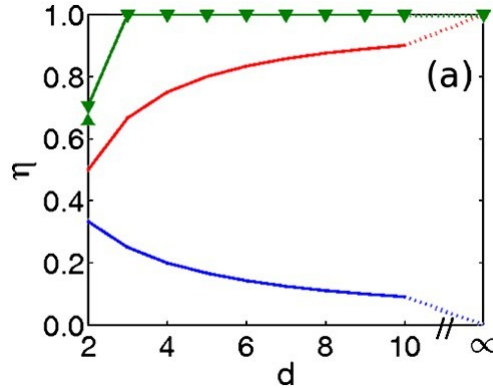


Figure 2. Parameter η (see text) boundaries for Werner States. The blue line corresponds to $\eta = 1/(d+1)$, the red line to $\eta = (1 - d^{-1})$ and the green line to $\eta = 1$ for $d \geq 3$. For η below blue line the states are Category 1 - separable states. These states are also Bell local, LHS and non-steerable. For η between blue line and red line the states are Category 2. These states are also Bell local, non-steerable and entangled. For η between red line and green line the states are Category 3 - Bell local, steerable and entangled (EPR entangled).

For η above green line the states are Category 4 - Bell non-local, steerable and entangled. Figure taken from Wiseman et al Ref [5].

12 Appendix D - Mean Values and Variances - General Features

12.1 Mean Values and Variances - Quantum Models

In a fully quantum treatment, any observable represented by a hermitian operator $\hat{\Omega}$ - whose measured outcomes are its eigenvalues θ , can be written as $\hat{\Omega} = \sum_{\theta} \theta \hat{\Pi}_{\theta}$ in terms of its projectors $\hat{\Pi}_{\theta}$ and we can determine the probability

$P(\hat{\Omega}, \theta)$ for the outcome θ via $P(\hat{\Omega}, \theta) = \text{Tr}(\hat{\Pi}_{\theta} \hat{\rho})$ - where $\hat{\rho}$ is the density operator that specifies the quantum state. Hence the mean value of the measured outcomes can be defined and then determined as follows

$$\langle \hat{\Omega} \rangle_Q = \sum_{\theta} \theta P(\hat{\Omega}, \theta) \quad (146)$$

$$= \text{Tr}(\hat{\Omega} \hat{\rho}) \quad (147)$$

We can also extend the concept of the mean value for measured outcomes to the case of a non-Hermitian operator $\hat{\Omega}$ - which although it does not correspond to an observable can be written in the form $\hat{\Omega} = \hat{\Omega}_1 + i\hat{\Omega}_2$, where both $\hat{\Omega}_1$ and $\hat{\Omega}_2$ are each observable Hermitian operators, not necessarily commuting. We simply define the mean for $\hat{\Omega}$ via

$$\begin{aligned} \langle \hat{\Omega} \rangle &\equiv \langle \hat{\Omega}_1 \rangle + i \langle \hat{\Omega}_2 \rangle \\ &= \text{Tr}(\hat{\Omega}_1 + i\hat{\Omega}_2) \hat{\rho} \end{aligned} \quad (148)$$

where $\langle \hat{\Omega}_1 \rangle$ and $\langle \hat{\Omega}_2 \rangle$ are defined as in (146), and we see that the result is given by the trace process. This definition and result can be applied to provide a meaning for the quantum mean values of operators such as an annihilation operator $\hat{a} = \frac{1}{\sqrt{2}}(\hat{x}_A + i\hat{p}_A)$ - which can be written in terms of *quadrature operators* or a transition operator $\hat{b}^\dagger \hat{a} = \hat{S}_x + i\hat{S}_y$ - which can be expressed in terms of *spin operators*. The latter case applies for considering *correlation tests*. If $\hat{\Omega}$ can be written as the sum of products of Hermitian sub-system operators $\hat{\Omega}_A$ and $\hat{\Omega}_B$ the last expression can be used to evaluate the mean value based on the quantum probability distributions for measurements of each $\hat{\Omega}_A$ and $\hat{\Omega}_B$.

Note that in expressing $\langle \hat{\Omega} \rangle$ in terms of $\langle \hat{\Omega}_1 \rangle$ and $\langle \hat{\Omega}_2 \rangle$ we are considering the results of two *independent* sets of measurements, one set for $\hat{\Omega}_1$ and the other for $\hat{\Omega}_2$. We do not imply that there is a joint probability $P(\omega_1, \omega_2 | \Omega_1, \Omega_2, c)$ for simultaneous outcomes ω_1, ω_2 of a combined measurement of Ω_1, Ω_2 following preparation c . We only require *single* measurement probabilities $P(\omega_1 | \Omega_1, c)$ and $P(\omega_2 | \Omega_2, c)$ to exist in order to define the mean values via $\langle \hat{\Omega}_1 \rangle = \sum_{\omega_1} \omega_1 P(\omega_1 | \Omega_1, c)$.

- which corresponds to the set of measurements on $\hat{\Omega}_1$ *alone*. In von-Neumann's proof that hidden variable theories were *inconsistent* with quantum theory, he

had evidently used the equivalent of $\langle \hat{\Omega} \rangle = \sum_{\omega_1} \sum_{\omega_2} (\omega_1 + i\omega_2) P(\omega_1, \omega_2 | \Omega_1, \Omega_2, c)$

based on *one* set of measurements, whereas we just use $\langle \hat{\Omega} \rangle = \sum_{\omega_1} (\omega_1) P(\omega_1 | \Omega_1, c) + i \sum_{\omega_2} (\omega_2) P(\omega_2 | \Omega_2, c)$ - which rests on two independent sets of measurements.

In the case of quantum separable states the *mean values* for jointly measuring Ω_A in sub-system A and Ω_B in sub-system B for preparation ρ would be given by

$$\langle \Omega_A \Omega_B \rangle = \sum_R P_R \langle \Omega_A \rangle_R \langle \Omega_B \rangle_R \quad (149)$$

where $\langle \Omega_A \rangle_R = \sum_{\alpha} \alpha P_Q(\alpha | \Omega_A, \rho, R) = \text{Tr}(\hat{\Omega}_A \hat{\rho}_R^A)$ and $\langle \Omega_B(\lambda) \rangle_Q = \sum_{\beta} \beta P_Q(\beta | \Omega_B, \rho, R) =$

$\text{Tr}(\hat{\Omega}_B \hat{\rho}_R^B)$ are the mean values for measurement outcomes for Ω_A and Ω_B . For the quantum separable state the mean value for *any* sum of products of subsystem operators which is Hermitian overall would be given by

$$\left\langle \sum_i \hat{\Omega}_{Ai} \hat{\Omega}_{Bi} \right\rangle = \sum_R P_R \sum_i \langle \hat{\Omega}_{Ai} \rangle_R \langle \hat{\Omega}_{Bi} \rangle_R \quad (150)$$

where $\langle \hat{\Omega}_{Ai} \rangle_R = \text{Tr}(\hat{\Omega}_{Ai} \hat{\rho}_R^A)$ and $\langle \hat{\Omega}_{Bi} \rangle_R = \text{Tr}(\hat{\Omega}_{Bi} \hat{\rho}_R^B)$ are quantum mean values, since we can always write $\hat{\Omega}_{Ai} = \hat{\Omega}_{Ai}^{(1)} + i\hat{\Omega}_{Ai}^{(2)}$ where both $\hat{\Omega}_{Ai}^{(1)}$ and $\hat{\Omega}_{Ai}^{(2)}$ are Hermitian and can be regarded as observables so with $\hat{\Omega}_{Ai} \hat{\Omega}_{Bi} = \hat{\Omega}_{Ai}^{(1)} \hat{\Omega}_{Bi}^{(1)} - \hat{\Omega}_{Ai}^{(2)} \hat{\Omega}_{Bi}^{(2)} + i(\hat{\Omega}_{Ai}^{(1)} \hat{\Omega}_{Bi}^{(2)} - \hat{\Omega}_{Ai}^{(2)} \hat{\Omega}_{Bi}^{(1)})$ which is of the form $\hat{\Omega}_1 + i\hat{\Omega}_2$, where both $\hat{\Omega}_1$ and $\hat{\Omega}_2$ are each observable Hermitian operators (the A and B operators commute). We can then invoke the probability distributions for the $\hat{\Omega}_{Ai}^{(1)}$, $\hat{\Omega}_{Bi}^{(1)}$, $\hat{\Omega}_{Ai}^{(2)}$ and $\hat{\Omega}_{Bi}^{(2)}$ to derive the expression for the mean value of $\hat{\Omega}_{Ai} \hat{\Omega}_{Bi}$ by also using (148). So (150) applies even if quantum operators $\hat{\Omega}_{Ai}$ and $\hat{\Omega}_{Bi}$ do not represent observables.

Variances can be obtained based on considering the mean values of the square of $\hat{\Omega}$. For observable represented by a hermitian operator $\hat{\Omega}$ the variance is defined by the mean of the squared variation of outcomes from the mean and equal to the difference between the mean of $\hat{\Omega}^2$ and the square of the mean of $\hat{\Omega}$.

$$\begin{aligned} \langle \Delta \hat{\Omega}^2 \rangle_Q &= \sum_{\theta} (\theta - \langle \hat{\Omega} \rangle_Q)^2 P(\hat{\Omega}, \theta) \\ &= \langle \hat{\Omega}^2 \rangle_Q - \langle \hat{\Omega} \rangle_Q^2 \end{aligned} \quad (151)$$

In the case of a *mixed* state (such as the QSS)

$$\hat{\rho} = \sum_R P_R \hat{\rho}_R \quad (152)$$

the *mean* for a Hermitian operator $\hat{\Omega}$ in a mixed state is the average of means for separate components

$$\langle \hat{\Omega} \rangle = \sum_R P_R \langle \hat{\Omega} \rangle_R \quad (153)$$

where $\langle \hat{\Omega} \rangle_R = \text{Tr}(\hat{\rho}_R \hat{\Omega})$. The variance for a Hermitian operator $\hat{\Omega}$ in a mixed state is always never less than the the average of the variances for the separate components (see [22])

$$\langle \Delta \hat{\Omega}^2 \rangle \geq \sum_R P_R \langle \Delta \hat{\Omega}_R^2 \rangle \quad (154)$$

where $\langle \Delta \hat{\Omega}^2 \rangle = \text{Tr}(\hat{\rho} \Delta \hat{\Omega}^2)$ with $\Delta \hat{\Omega} = \hat{\Omega} - \langle \hat{\Omega} \rangle$ and $\langle \Delta \hat{\Omega}^2 \rangle_R = \text{Tr}(\hat{\rho}_R \Delta \hat{\Omega}_R^2)$ with $\Delta \hat{\Omega}_R = \hat{\Omega} - \langle \hat{\Omega} \rangle_R$. To prove this result we have using (153) both for $\hat{\Omega}$ and $\hat{\Omega}^2$

$$\begin{aligned} \langle \Delta \hat{\Omega}^2 \rangle &= \langle \hat{\Omega}^2 \rangle - \langle \hat{\Omega} \rangle^2 \\ &= \sum_R P_R \left(\langle \hat{\Omega}^2 \rangle_R - \langle \hat{\Omega} \rangle_R^2 \right) + \sum_R P_R \langle \hat{\Omega} \rangle_R^2 - \left(\sum_R P_R \langle \hat{\Omega} \rangle_R \right)^2 \\ &= \sum_R P_R \langle \Delta \hat{\Omega}_R^2 \rangle + \sum_R P_R \langle \hat{\Omega} \rangle_R^2 - \left(\sum_R P_R |\langle \hat{\Omega} \rangle_R| \right)^2 \end{aligned} \quad (155)$$

The variance result (154) follows because the sum of the last two terms is always ≥ 0 using the result (135) in Appendix E of Ref [2], with $C_R = \langle \hat{\Omega} \rangle_R^2$, $\sqrt{C_R} = |\langle \hat{\Omega} \rangle_R|$ - which are real and positive.

In considering the means and variances in the context of LHV several difficult issues need to be dealt with. Firstly, in a LHV the observables are basically considered as classical c-numbers, but given that the predictions from quantum theory are accepted as being correct these classical observables must correspond to underlying quantum Hermitian operators - especially as in the LHS model where the probabilities $P_Q(\beta|\Omega_B, c, \lambda)$ for subsystem B are also to be given by quantum formulae. Also, there are several entanglement tests involving spin components, these are represented by the spin operators $\hat{S}_x = (\hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b})/2$, $\hat{S}_y = (\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b})/2i$ and $\hat{S}_z = (\hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{a})/2$, where \hat{a} and \hat{b} are mode annihilation operators. The tests also involve the total number operator $\hat{N} = (\hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a})$. All these operators are Hermitian and represent observable quantities applying for the overall two mode system. We may also consider number operators for the two modal subsystems defined by $\hat{N}_A = \hat{a}^\dagger \hat{a}$ and $\hat{N}_B = \hat{b}^\dagger \hat{b}$, which again are Hermitian and represent observable quantities for each subsystem. The question then arises: How do you define the spin components and the boson number when the observables are supposed to be non-quantum? Secondly, when considering

entanglement tests involving spin components, both subsystem A and B involve mode annihilation operators - which are non-Hermitian and not themselves associated with measurable observables. What meaning can we give to LHV probabilities $P(\alpha|\Omega_A, c, \lambda)$ and associated mean values $\langle \Omega_A(\lambda) \rangle = \sum_{\alpha} \alpha P(\alpha|\Omega_A, c, \lambda)$

for subsystem A when during the discussion of spin squeezing tests in the LHS model we consider situations where Ω_A corresponds to a mode annihilation or creation operator ? To give meaning to the LHS model do we need to consider LHV probabilities $P(\alpha_1, \alpha_2|\Omega_{A1}, \Omega_{A2}, c, \lambda)$ associated with the outcomes of measuring *two* observables Ω_{A1}, Ω_{A2} for sub-system A when the hidden variables are λ and which may correspond to quantum operators that do not commute ? What happens when we need to consider a product such as $\Omega_{A1}\Omega_{A2}\Omega_{B1}\Omega_{B2}$ such as may occur when we are considering expressions for variances ? Would this mean that for products of subsystem observables we should determine the mean values via

$$\langle \Omega_{A1}\Omega_{A2}\Omega_{B1}\Omega_{B2} \rangle = \sum_{\lambda} P(\lambda|c) \langle \Omega_{A1}\Omega_{A2}(\lambda) \rangle \langle \Omega_{B1}\Omega_{B2}(\lambda) \rangle_Q$$

where $\langle \Omega_{A1}\Omega_{A2}(\lambda) \rangle = \sum_{\alpha_1, \alpha_2} \alpha_1 \alpha_2 P(\alpha_1, \alpha_2|\Omega_{A1}, \Omega_{A2}, c, \lambda)$ and $\langle \Omega_{B1}\Omega_{B2}(\lambda) \rangle_Q =$

$\sum_{\beta_1, \beta_2} \beta_1 \beta_2 P_Q(\beta_1, \beta_2|\Omega_{B1}, \Omega_{B2}, c, \lambda)$? But what meaning is there to the quantum expression when the corresponding operators $\hat{\Omega}_{B1}, \hat{\Omega}_{B2}$ do not commute ?

None of these questions arise in considering whether spin squeezing is a test for standard quantum entanglement, since no hidden variables are involved nor are issues of the existence of probabilities for measurement of individual subsystem operators that may become involved in the evaluation. However, when non-quantum LHV expressions for measurement probabilities are involved, the analogous results to those for quantum mean values need further consideration. Until these issues are resolved we cannot begin to modify the operator based proof regarding the consequences for spin variances and means for the LHS state. The proof would involve expressions giving meaningful interpretations to the mean values of what would appear to be non-physical quantities such as mode annihilation and creation operators for subsystem A .

12.2 General Results for Mean and Variance in LHV Theory

Before dealing with the above issues it is useful to prove some results for mean values and variances in LHV that are analogous to similar results in quantum theory. We now consider the measurement of an observable Ω with outcomes ω for a preparation process c . The probability $P(\omega|\Omega, c)$ for this outcome can be

written in LHV as

$$P(\omega|\Omega, c) = \sum_{\lambda} P(\lambda|c) P(\omega|\Omega, c, \lambda) \quad (156)$$

where λ are the hidden variables and $P(\lambda|c)$ is the probability for preparation process c that the hidden variables are λ and $P(\omega|\Omega, c, \lambda)$ is the probability of outcome ω for measurement of Ω when the hidden variables are λ .

The *mean value* for measurement outcomes for observable Ω will then be given by

$$\langle \Omega \rangle = \sum_{\omega} \omega P(\omega|\Omega, c) \quad (157)$$

$$= \sum_{\lambda} P(\lambda|c) \langle \Omega(\lambda) \rangle \quad (158)$$

$$\langle \Omega(\lambda) \rangle = \sum_{\omega} \omega P(\omega|\Omega, c, \lambda) \quad (159)$$

where the first equation is the definition and the second equation shows that the mean value is given by weighting the mean value $\langle \Omega(\lambda) \rangle$ that would apply if the hidden variables are λ by the probability $P(\lambda|c)$ for these hidden variables when the preparation is c . The result (158) is similar to the quantum result for the mixed state $\hat{\rho} = \sum_R P_R \hat{\rho}_R$ where $\langle \hat{\Omega} \rangle = \sum_R P_R \langle \hat{\Omega} \rangle_R$ where $\langle \hat{\Omega} \rangle_R = \text{Tr}(\hat{\Omega} \hat{\rho}_R)$.

The result for the mean value of a *function* $F(\Omega)$ would be

$$\begin{aligned} \langle F(\Omega) \rangle &= \sum_{\lambda} P(\lambda|c) \langle F(\Omega)_{\lambda} \rangle \\ \langle F(\Omega)_{\lambda} \rangle &= \sum_{\omega} F(\omega) P(\omega|\Omega, c, \lambda) \end{aligned} \quad (160)$$

In the case where the outcomes for *two* observables Ω and Λ with outcomes ω and μ for a preparation process c , the mean value for a *function* $F(\Omega, \Lambda)$ would be

$$\begin{aligned} \langle F(\Omega, \Lambda) \rangle &= \sum_{\lambda} P(\lambda|c) \langle F(\Omega, \Lambda)_{\lambda} \rangle \\ \langle F(\Omega, \Lambda)_{\lambda} \rangle &= \sum_{\omega\mu} F(\omega, \mu) P(\omega, \mu|\Omega, \Lambda, c, \lambda) \end{aligned} \quad (161)$$

This result will be useful when we consider steering tests.

The *variance* for measurement outcomes for observable Ω will then be given by

$$\langle \Delta \Omega^2 \rangle = \sum_{\omega} (\omega - \langle \Omega \rangle)^2 P(\omega|\Omega, c) \quad (162)$$

$$\begin{aligned} &= \sum_{\omega} (\omega^2 - 2\omega \langle \Omega \rangle + \langle \Omega \rangle^2) P(\omega|\Omega, c) \\ &= \langle \Omega^2 \rangle - \langle \Omega \rangle^2 \end{aligned} \quad (163)$$

$$\langle \Omega^2 \rangle = \sum_{\omega} \omega^2 P(\omega|\Omega, c) \quad (164)$$

where the first equation is the definition and the third equation shows that the variance is given by the difference between the mean of the squared observable and the square of the mean, as in standard statistics. Here we have used $\sum_{\omega} P(\Omega|\omega, c) = 1$ and (157). We can then write

$$\langle \Omega^2 \rangle = \sum_{\lambda} P(\lambda|c) \langle \Omega^2(\lambda) \rangle \quad (165)$$

$$\langle \Omega^2(\lambda) \rangle = \sum_{\omega} \omega^2 P(\omega|\Omega, \lambda, c) \quad (166)$$

where the second line gives the definition for the mean of the square of the observable when the hidden variables are λ and the first line expresses the mean of the square of the observable in terms of an average over this quantity.

We then have

$$\begin{aligned} \langle \Delta \Omega^2 \rangle &= \sum_{\lambda} P(\lambda|c) \langle \Omega^2(\lambda) \rangle - (\sum_{\lambda} P(\lambda|c) \langle \Omega(\lambda) \rangle)^2 \\ &\geq \sum_{\lambda} P(\lambda|c) (\langle \Omega^2(\lambda) \rangle - \langle \Omega(\lambda) \rangle^2) + \sum_{\lambda} P(\lambda|c) \langle \Omega(\lambda) \rangle^2 - (\sum_{\lambda} P(\lambda|c) |\langle \Omega(\lambda) \rangle|)^2 \\ &\geq \sum_{\lambda} P(\lambda|c) (\langle \Omega^2(\lambda) \rangle - \langle \Omega(\lambda) \rangle^2) \end{aligned} \quad (167)$$

which establishes an important inequality. The second line follows from the modulus of a sum being less than the sum of the moduli and the last line follows from the Cauchy inequality $\sum_R P_R C_R \geq (\sum_R P_R \sqrt{C_R})^2$ with $\sqrt{C_R} = |\langle \Omega(\lambda) \rangle|$.

But we also have

$$\langle \Delta \Omega^2(\lambda) \rangle = \sum_{\omega} (\omega - \langle \Omega(\lambda) \rangle)^2 P(\omega|\Omega, c, \lambda) \quad (168)$$

$$\begin{aligned} &= \sum_{\omega} \omega^2 P(\omega|\Omega, c, \lambda) - \langle \Omega(\lambda) \rangle^2 \\ &= \langle \Omega^2(\lambda) \rangle - \langle \Omega(\lambda) \rangle^2 \end{aligned} \quad (169)$$

showing that when the hidden variable is λ the variance for measured outcomes of observable Ω is equal to the difference between the mean value for measured outcomes of the square of the observable and the square of the mean value (as expected).

We finally have the inequality

$$\langle \Delta \Omega^2 \rangle \geq \sum_{\lambda} P(\lambda|c) \langle \Delta \Omega^2(\lambda) \rangle \quad (170)$$

This result may be compared to the quantum theory result $\langle \Delta \hat{\Omega}^2 \rangle \geq \sum_R P_R \langle \Delta \hat{\Omega}^2 \rangle_R$.

Finally, we consider mean values in LHV for complex combinations of observables Ω_1 and Ω_2 , which have measured outcomes ω_1 and ω_2 . We simply define

$$\langle (\Omega_1 + i\Omega_2) \rangle = \langle \Omega_1 \rangle + i \langle \Omega_2 \rangle \quad (171)$$

where in HVT we have

$$\begin{aligned}\langle \Omega_1 \rangle &= \sum_{\lambda} P(\lambda|c) \sum_{\omega_1} \omega_1 P(\Omega_1|\omega_1, c, \lambda) \\ \langle \Omega_2 \rangle &= \sum_{\lambda} P(\lambda|c) \sum_{\omega_2} \omega_2 P(\Omega_2|\omega_2, c, \lambda)\end{aligned}\quad (172)$$

since the two fundamental probabilities $P(\omega_1|\Omega_1, c, \lambda)$ and $P(\omega_2|\Omega_2, c, \lambda)$ always exist in a LHV, *even if* in quantum theory the corresponding operators $\hat{\Omega}_1$ and $\hat{\Omega}_2$ do *not* commute. This is an important feature to recognise about LHV. The result (171) may be compared to the quantum result (148). Thus, we see that many results in LHV are analogous to the results in quantum theory.

With these results now established we can see that for the LHS model the *mean values* for jointly measuring Ω_A in sub-system A and Ω_B in sub-system B for preparation c would be given by

$$\langle \Omega_A \otimes \Omega_B \rangle = \sum_{\lambda} P(\lambda|c) \langle \Omega_A(\lambda) \rangle \langle \Omega_B(\lambda) \rangle_Q \quad (173)$$

where $\langle \Omega_A(\lambda) \rangle = \sum_{\alpha} \alpha P(\alpha|\Omega_A, c, \lambda)$ and $\langle \Omega_B(\lambda) \rangle_Q = \sum_{\beta} \beta P_Q(\beta|\Omega_B, c, \lambda) =$

$Tr(\hat{\Omega}_B \hat{\rho}_{\lambda}^B)$ are the definitions of the mean values for measurement outcomes for Ω_A and Ω_B . The latter is also determined from quantum theory, the former is not. *Variances* can be obtained based on considering the mean values of the squares of Ω_A and Ω_B . The similarities and differences between the LHS and the QSS expressions (173) and (149) should be noted.

13 Appendix E - Difference in Variances of Number Sum and Difference - Category 2 States ?

The experimental paper by Gross et al [23] involving atomic homodyne detection in a two mode system in which the two modes (signal and idler) are populated from a BEC with all atoms in an $m = 0$ hyperfine state. Collisions of two $m = 0$ hyperfine state atoms lead to one atom in each of $m = +1$ and $m = -1$ hyperfine states, corresponding to the signal and idler modes. Measurements of the variance for the *number difference* and *number sum* for the signal and idler modes show a small fluctuation for the number difference and a large fluctuation for the number sum. The experiment also measures two mode *quadrature* variances and the results are interpreted as demonstrating EPR steering. However, another possibility worth exploring is whether the difference between the variances of the number sum and number difference observables might itself provide a test for EPR steering. As we will see, no such test has been found.

We now consider the difference between the variances of the number sum and number difference observables. From (51) the number sum and number difference quantum operators are

$$\begin{aligned}\hat{N}_+ &= \hat{N}_B + \hat{N}_A = \hat{N} \\ \hat{N}_- &= \hat{N}_B - \hat{N}_A = 2\hat{S}_z\end{aligned}\tag{174}$$

so that the classical observables are then given by

$$\begin{aligned}N_+ &= N_B + N_A \\ N_- &= N_B - N_A\end{aligned}\tag{175}$$

where N_A and N_B are given by (51) in terms of quadrature amplitudes. The question is whether a test for EPR steering can be obtained based on the differences in the variances of N_+ and N_- .

The differences between the variances for N_+ and N_- are given in a LHV via

$$\begin{aligned}&\langle \Delta N_+^2 \rangle - \langle \Delta N_-^2 \rangle \\&= \langle (N_B + N_A)^2 \rangle - \langle (N_B + N_A) \rangle^2 - \langle (N_B - N_A)^2 \rangle + \langle (N_B - N_A) \rangle^2 \\&= \langle N_B^2 \rangle + \langle N_A^2 \rangle + \langle N_B N_A \rangle + \langle N_A N_B \rangle - \langle N_B \rangle^2 - \langle N_A \rangle^2 - 2 \langle N_A \rangle \langle N_B \rangle \\&\quad - \langle N_B^2 \rangle - \langle N_A^2 \rangle + \langle N_B N_A \rangle + \langle N_A N_B \rangle + \langle N_B \rangle^2 + \langle N_A \rangle^2 - 2 \langle N_A \rangle \langle N_B \rangle \\&= 4 (\langle N_A N_B \rangle - \langle N_A \rangle \langle N_B \rangle)\end{aligned}\tag{176}$$

For the *LHS model* we then find that

$$\begin{aligned}
\langle N_A N_B \rangle &= \sum_{\lambda} P(\lambda|c) \langle N_A(\lambda) \rangle \langle N_B(\lambda) \rangle_Q \\
\langle N_A \rangle &= \sum_{\lambda} P(\lambda|c) \langle N_A(\lambda) \rangle \\
\langle N_B \rangle &= \sum_{\lambda} P(\lambda|c) \langle N_B(\lambda) \rangle_Q
\end{aligned} \tag{177}$$

Finally then after reverting to the quantum theory expressions using (25) and (26) we have

$$\begin{aligned}
&\langle \Delta N_+^2 \rangle - \langle \Delta N_-^2 \rangle \\
&= 4 \left(\sum_{\lambda} P(\lambda|c) \langle N_A(\lambda) \rangle \langle N_B(\lambda) \rangle_Q - \sum_{\lambda} P(\lambda|c) \langle N_A(\lambda) \rangle \sum_{\mu} P(\mu|c) \langle N_B(\mu) \rangle_Q \right) \\
&= 4 \left(\langle \hat{N}_A \otimes \hat{N}_B \rangle - \langle \hat{1}_A \otimes \hat{N}_B \rangle \langle \hat{N}_A \otimes \hat{1}_B \rangle \right) \\
&= 4 \left(\sum_{n_A n_B} n_A n_B \rho_{n_A n_B ; n_A n_B} - \left\{ \sum_{n_A n_B} n_A \rho_{n_A n_B ; n_A n_B} \right\} \left\{ \sum_{m_A m_B} m_B \rho_{m_A m_B ; m_A m_B} \right\} \right)
\end{aligned} \tag{178}$$

An inequality is needed for the right side. The Cauchy inequality $\sum_R P_R C_R \sum_S P_S D_S \geq (\sum_R P_R \sqrt{C_R D_R})^2$ with $R \rightarrow n_A n_B$, $P_R \rightarrow \rho_{n_A n_B ; n_A n_B}$, $C_R \rightarrow n_A$ and $D_R \rightarrow n_B$ is applicable since we know that both n_A and n_B are positive and $\sum_{n_A n_B} \rho_{n_A n_B ; n_A n_B} = 1$. We see that

$$\begin{aligned}
\langle \Delta N_+^2 \rangle - \langle \Delta N_-^2 \rangle &\leq 4 \left(\sum_{n_A n_B} n_A n_B \rho_{n_A n_B ; n_A n_B} - \left\{ \sum_{n_A n_B} \sqrt{n_A n_B} \rho_{n_A n_B ; n_A n_B} \right\}^2 \right) \\
&\leq 4 \left(\langle (\sqrt{n_A n_B})^2 \rangle - \langle \sqrt{n_A n_B} \rangle^2 \right) \\
&\leq 4 \langle \Delta(\sqrt{n_A n_B})^2 \rangle
\end{aligned} \tag{179}$$

The right side of (179) is always ≥ 0 since the mean of a square always exceeds the square of the mean. But this is consistent with

$$\langle \Delta N_+^2 \rangle - \langle \Delta N_-^2 \rangle > 0 \quad \text{or} \quad \langle \Delta N_+^2 \rangle - \langle \Delta N_-^2 \rangle < 0 \tag{180}$$

so we do not have a test for EPR entanglement since $\langle \Delta N_+^2 \rangle - \langle \Delta N_-^2 \rangle$ may be negative or positive for the LHS model.

Hence *no* test based on the differences of variances for the number sum and number difference is available for *EPR steering*.

14 Appendix F - EPR Steering Test - Other Approach

14.1 Details of EPR Test Derivation

The inequalities (108), (110) and (112) derived above can be put into a more useful form involving *spin operators* - whose mean values and variances can be measured. We use the definitions of the spin operators in SubSection 5.2 (see also Ref. [3])

$$\begin{aligned}
 |\langle \hat{a}^\dagger \hat{b} \rangle|^2 &= \langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2 \\
 \hat{N}_A &= \frac{1}{2} \hat{N} - \hat{S}_z & \hat{N}_B &= \frac{1}{2} \hat{N} + \hat{S}_z \\
 \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 &= \frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right)
 \end{aligned} \tag{181}$$

We see that

$$\begin{aligned}
 \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle &= \frac{1}{4} \langle (\hat{N}_A + \hat{N}_B)^2 \rangle + \frac{1}{2} \langle \hat{N}_A + \hat{N}_B \rangle - |\langle \hat{a}^\dagger \hat{b} \rangle|^2 - \frac{1}{4} \langle (\hat{N}_B - \hat{N}_A)^2 \rangle \\
 &= \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle - |\langle \hat{a}^\dagger \hat{b} \rangle|^2 \\
 &\geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \langle \hat{N}_A \otimes \hat{N}_B \rangle \quad \text{Cat 1 States} \\
 &\geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle \quad \text{Cat 1 States} \\
 &\geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \left\langle \left(\hat{N}_A + \frac{1}{2} \hat{1}_A \right) \otimes \hat{N}_B \right\rangle \quad \text{Cat 2 States} \\
 &\geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle \quad \text{Cat 2 States} \\
 &\geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \left\langle \left(\hat{N}_A + \frac{1}{2} \hat{1}_A \right) \otimes \left(\hat{N}_B + \frac{1}{2} \hat{1}_B \right) \right\rangle \\
 &\geq -\frac{1}{4} \quad \text{Cat 3 States}
 \end{aligned}$$

So we have

$$\begin{aligned}
 \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle &\geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle - \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle \\
 &\geq 0 \quad \text{Cat 1 States} \\
 \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle &\geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle - \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle - \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle \\
 &\quad - \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle \\
 &\geq 0 \quad \text{Cat 2 States} \\
 \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle + \frac{1}{4} &\geq -\frac{1}{4} + \frac{1}{4} \\
 &\geq 0 \quad \text{Cat 3 States}
 \end{aligned} \tag{183}$$

14.2 Another Derivation of EPR Test

The EPR steering test in (119) can also be confirmed using the results in Section 5.5. Using (78), (80) and (79) we find for Category 2 states

$$\begin{aligned} \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle &\geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle \\ &\geq 0 \end{aligned} \quad (184)$$

Details are:

$$\begin{aligned} \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle &\geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{1}_B \rangle \\ &\quad - \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle \\ &\quad + \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle \\ &\geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle \\ &\geq 0 \end{aligned}$$

since both $\langle \hat{N}_A \otimes \hat{N}_B \rangle$ and $\langle \hat{1}_A \otimes \hat{N}_B \rangle$ are positive quantities. The previous approach produced a smaller right side to the inequality than $\langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle$, but even in this form it shows that if $\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle < 0$ then the state cannot be Category 2.